New Calabi–Yau Manifolds from Genetic Algorithms

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Goal

Generate Calabi-Yau fourfolds suitable for F-theory model building.

Method

Use a genetic algorithm to generate 5-dimensional reflexive polytopes.

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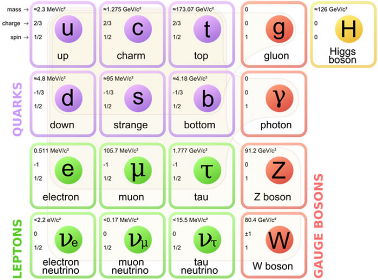
Outline

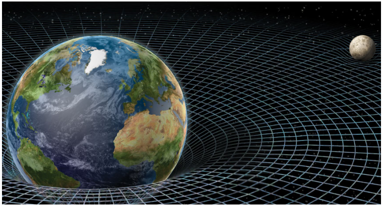
What is the state of physics in 2023 ?

Standard Model



The Standard Model of particle physics describes the three non-gravitational forces : strong nuclear, weak nuclear, and electromagnetic force - as well as all observed elementary particles.

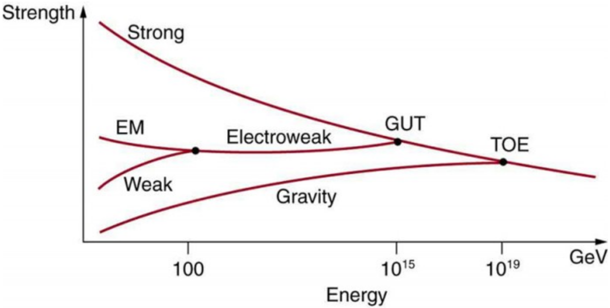
General Relativity

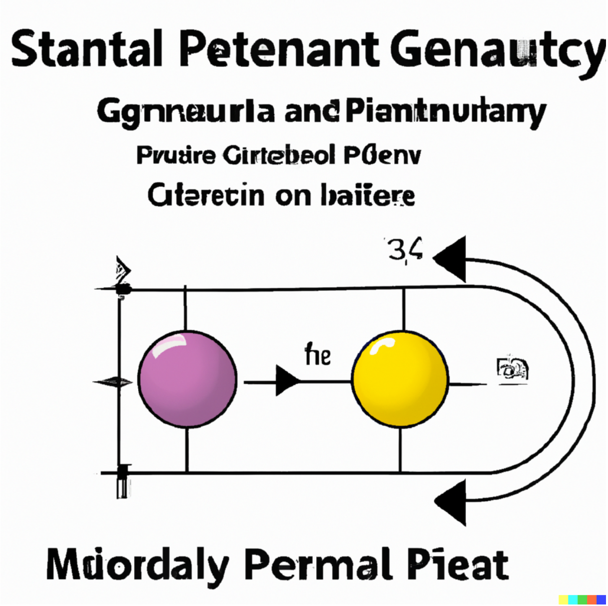


General relativity is a theoretical framework that only focuses on gravity and provides a description of gravity as a geometric property of space and time.

A Theory of Everything (TOE) is an all-encompassing theoretical framework of physics that fully explains all aspects of the universe.

A Grand Unified Theory (GUT) is a model in which, at high energies, the three gauge interactions of the Standard Model comprising the electromagnetic, weak, and strong forces are merged into a single force.

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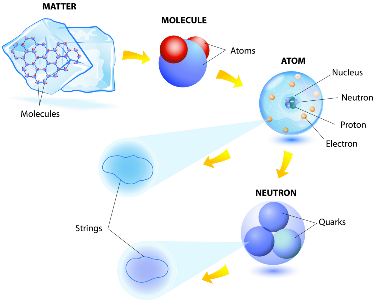
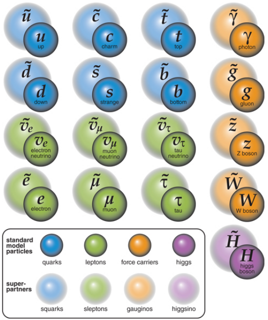
String theory is a theory in which 0-dimensional particles are replaced by 1-dimensional strings. The different vibrational modes of the string give rise to the different particle properties in the Standard Model.





Supersymmetry is a spacetime symmetry between two classes of particles : bosons and fermions. In supersymmetry, each particle from one class would have an associated particle in the other, known as its superpartner.

Problem

String theory only works in 10 dimensions of spacetime, but we live in 4 spacetime dimensions.

Solution

Hide the extra dimensions where no one can see them.

Compactification means that the 10-dimensions of spacetime are of the form :

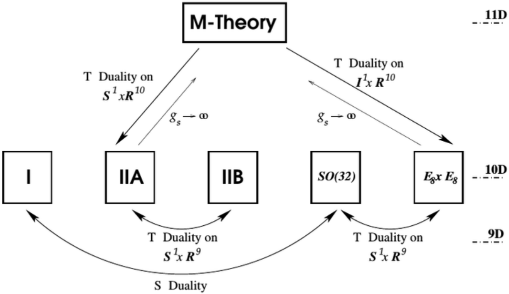
$$M_{10} = \mathbb{R}^{1,3} \times M_6,$$

where $\mathbb{R}^{1,3}$ is the usual Minkowski space and M_6 is some small compact manifold.

We must satisfy the vacuum Einstein equations, which means that the metric on M_6 must be **Ricci-flat**. Furthermore, in order to have the correct supersymmetry we also require that M_6 has **holonomy group** $SU(3)$.

Definition

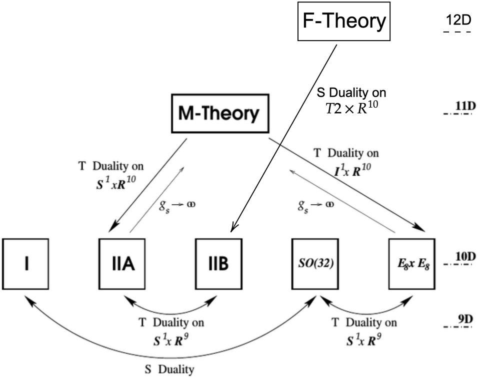
A Ricci-flat Kähler manifold with holonomy group $SU(3)$ is a Calabi-Yau manifold.



In order to get minimal $\mathcal{N} = 1$ 4d supergravity after compactification, one needs M-theory on 7-dimensional G2-manifolds. See Ed's talk for more on G2 manifolds.

$$M_{11} = \mathbb{R}^{1,3} \times G2$$

F-Theory

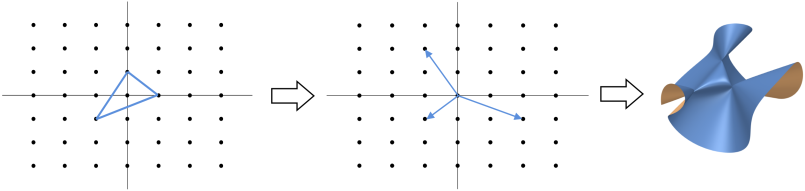


In order to get minimal $\mathcal{N} = 1$ 4d supergravity after compactification, one needs F-theory on elliptically fibered CY4-manifolds.

$$M_{12} = \mathbb{R}^{1,3} \times CY4$$

How to construct Calabi-Yau manifolds

Given an n-dimensional lattice polytope Δ, one can construct a toric variety X_Δ of complex dimension n. One constructs the normal fan Σ_Δ from the faces θ of Δ and glues together the affine toric varieties that arise from each cone σ_θ in the fan.



If Δ is reflexive it follows that the zero locus of a generic section of the anticanonical bundle $-K_X$ is a CY variety of dimension $n-1$, which is in general singular but whose singularities can be resolved by triangulating the polytope.

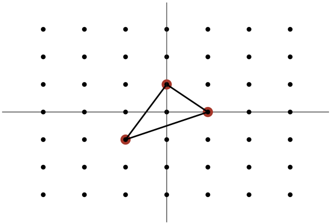
What are reflexive polytopes?

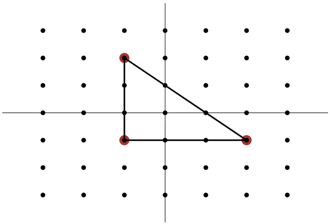
Let $M \cong \mathbb{Z}^n$ and $N = \mathrm{Hom}(M, \mathbb{Z})$ be a dual pair of lattices with the pairing $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$. Let Δ be a polytope in $M_{\mathbb{R}}$, then the **dual** polytope Δ^* in $N_{\mathbb{R}}$ is defined as

$$\Delta^* = \{n \in N_{\mathbb{R}} | \langle n, m \rangle \geq -1 \forall m \in \Delta\}.$$

A polytope is said to satisfy the interior point (**IP**) property when it contains only one interior point at the origin.

A lattice polytope Δ that satisfies the IP property is called **reflexive** if its dual Δ^* is also a lattice polytope satisfying the IP property.





Classification of reflexive polytopes

Algorithm by for generating all reflexive polytopes :

1. Construct a set $\mathcal{S}$ of maximal polytopes, such that any reflexive polytope is a subpolytope of a polytope in $\mathcal{S}$.
2. Construct all subpolytopes of the maximal polytopes in $\mathcal{S}$ and check for reflexivity.

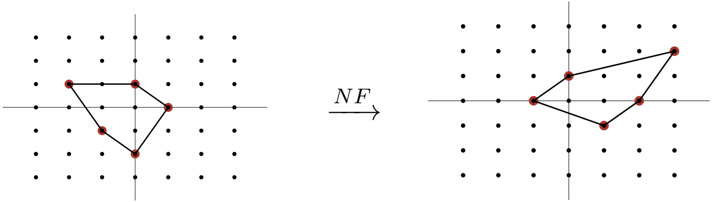
Classification of reflexive polytopes :

- There are 16 reflexive polytopes in 2d.
- There are 4,319 reflexive polytopes in 3d.
- There are 473,800,776 reflexive polytopes in 4d.
- There are **at least** 185,269,499,015 reflexive polytopes in 5d.

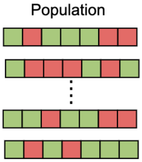
Two polytopes Δ and $\tilde{\Delta}$ are called equivalent if there exist an $N_{\mathrm{v}} \times N_{\mathrm{v}}$ permutation matrix P and a $G \in \mathrm{GL}(n, \mathbb{Z})$ such that their vertex matrices V and $\tilde{V}$ are related by

$$\tilde{V} = GVP.$$

The **normal form** is a representative of this equivalence class.



Genetic algorithms



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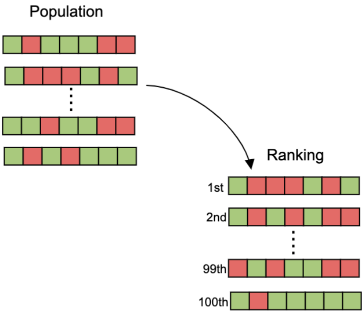
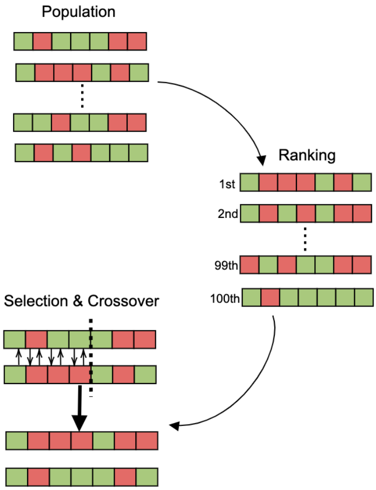
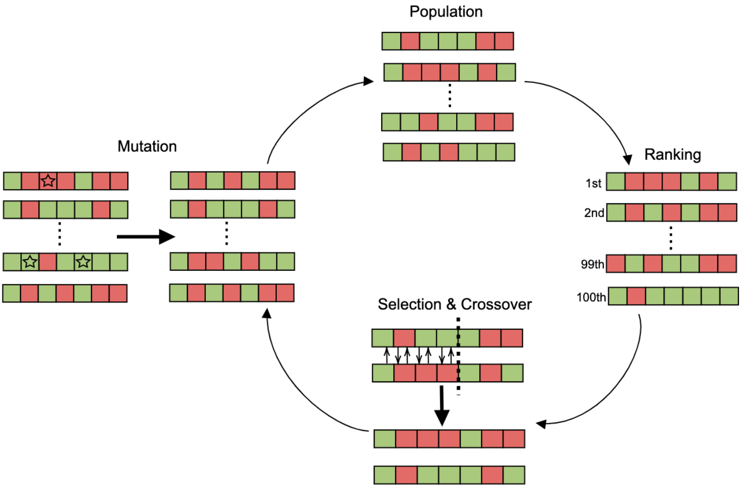


Genetic algorithms



Genetic algorithms



Environment

The GA environment in our case consists of lattice polytopes Δ in n dimensions which are generated as the convex hull of m vectors $x_a \in \mathbb{Z}^n$, where $a = 1, ..., m$. These vectors are arranged into an $n \times m$ matrix $X = (x_1, \ldots, x_m)$.

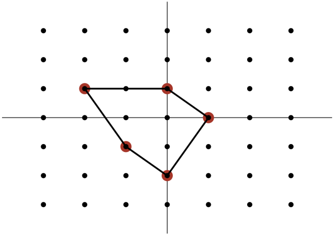
We convert the matrices X into a bitlists as follows :

1. Flatten the $n \times m$ vertex matrix X.
2. Subtract `MIN` from each entry.
3. Convert each integer entry into binary.
4. Prepend each binary number with zeros so that each binary number is of length `BINLEN`.

The environment is therefore the set $\mathbb{F}_2^{n_{\text{bits}}}$ of all bitlists of length n_{bits}, where $n_{\text{bits}} = m \times n \times \texttt{BINLEN}$, which is of size $2^{n_{\text{bits}}}$.

Environment

Example : `MIN`=-3, `BINLEN`=3



$$X = \begin{pmatrix} -2 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & -2 & -1 \end{pmatrix}$$

1. $[-2, 1, 0, 1, 1, 0, 0, -2, -1, -1]$
2. $[1, 4, 3, 4, 4, 3, 3, 1, 2, 2]$
3. $[1|1, 0, 0|1, 1|1, 0, 0|1, 0, 0|1, 1|1, 1|1|1, 0|1, 0]$
4. $[0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0]$

Fitness function

Let Δ be a lattice polytope given in the half-space representation by $\Delta = \{m \in M_{\mathbb{R}} | \langle u_i, m\rangle \leq a_i, i = 1, ..., k\}$ for some $u_1, ..., u_k \in N_{\mathbb{R}}$ and $a_1, ..., a_k \in \mathbb{R}$. Then Δ is reflexive if and only if it has the IP property and if $a_i = 1$ for all i.

The fitness function, $f : \mathbb{F}_2^{n_{\text{bits}}} \to \mathbb{R}$ is given by :

$$f(\Delta) = w_1\left(\mathrm{IP}(\Delta) - 1\right) - \frac{w_2}{k}\sum_{i=1}^{k}|a_i(\Delta) - 1| - w_3|N_{\mathrm{p}}(\Delta) - N_{\mathrm{p},0}|$$

- $\mathrm{IP}(\Delta)$ equals 1 if Δ satifies the IP property and is 0 otherwise.
- $N_{\mathrm{p}}(\Delta)$ is the number of points of Δ and $N_{\mathrm{p},0}$ is the desired number of points.
- $w_1, w_2, w_3 \in \mathbb{R}^{\geq 0}$ are weights.

Selection method

A probability distribution $p_k : P_k \to [0, 1]$, based on the fitness function, is computed for the k^{th} population P_k. The selection method we employed is the so-called roulette wheel selection where p_k for an individual $s \in P_k$ is defined by

$$p_k(s) = \frac{1}{n_{\text{pop}}} \frac{(\alpha - 1)\left(f(s) - \bar{f}\right) + f_{\max} - \bar{f}}{f_{\max} - \bar{f}} ,$$

where $\bar{f}$ and $f_{\max}$ are the average and maximal fitness values on P_k, respectively. The parameter α, typically chosen in the range $\alpha \in [2, 5]$, indicates by which factor the fittest individual in the population is more likely to be selected than the average one. Based on this probability p_k, $n_{\text{pop}}/2$ pairs are selected from the population P_k.

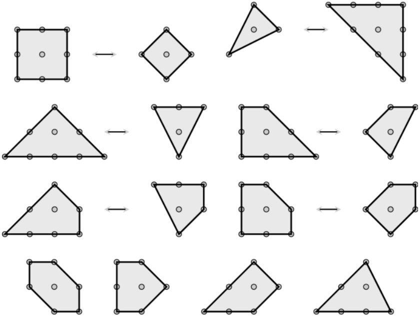
Process overview

1. Create a random population P_0 of $n_{\rm pop}$ bitlists.
2. Evolve the population over $n_{\rm gen}$ generations

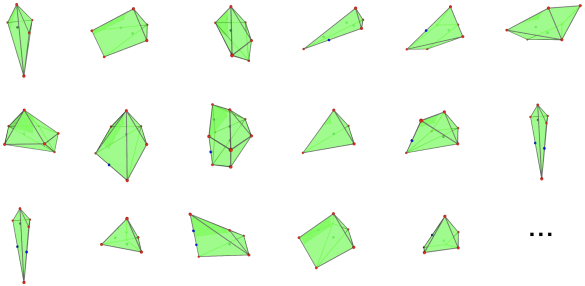
$$P_0 \to P_1 \to \cdots\cdots \to P_{n_{\rm gen}-1} \to P_{n_{\rm gen}}.$$

3. Extract all generated reflexive polytopes.
4. Remove redundancy in the list of reflexive polytopes by computing the normal forms and deleting duplicates.
5. Repeat steps 1-4 until all reflexive polytopes are found.

- We use `MIN` $= -3$ and `BINLEN` $= 3$ which gives us the vertex coordinate range is $[-3, 4]$ and set $m = 6$.
- The environment contains $\sim 10^{11}$ states.
- The genetic algorithm found all 16 reflexive polytopes in 1 run.
- The fraction of states visited was $\sim 10^{-6}$.



- We use $\texttt{MIN} = -7$ and $\texttt{BINLEN} = 4$ which gives us the vertex coordinate range is $[-7, 8]$ and set $m = 14$.
- The environment contains $\sim 10^{51}$ states.
- The genetic algorithm found all 4319 reflexive polytopes in 117251 runs.
- The fraction of states visited was $\sim 10^{-40}$.



Results : 4d

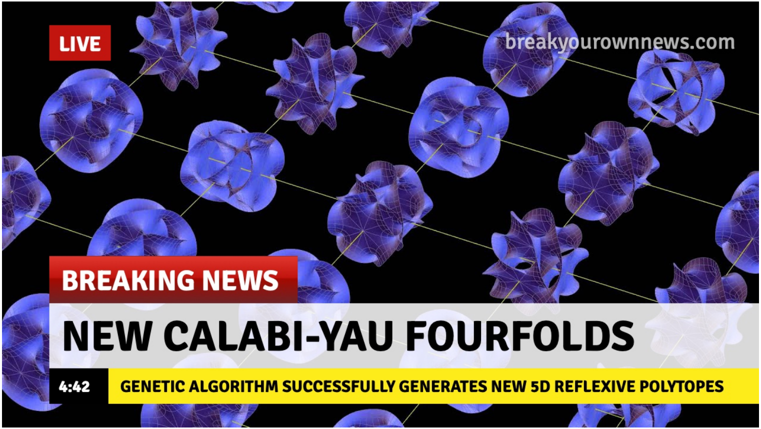
# points	# refl. poly.	# GA runs	% states
6	3	5	$\sim 10^{-13}$
7	25	30	$\sim 10^{-16}$
8	168	60	$\sim 10^{-19}$
9	892	9378	$\sim 10^{-20}$
10	3838	9593	$\sim 10^{-24}$

Table – Search results for four-dimensional reflexive polytopes. The number of lattice points is listed in the first column. The total number of generated reflexive polytopes is given in the second column and the number of genetic algorithm runs taken to reach this total is provided in the third column. The last column provides an upper bound on the fraction of states visiting during all GA runs.

Results : 5d

# points	# refl. poly.	# GA runs	% states
7	9	36	$\sim 10^{-22}$
8	115	1278	$\sim 10^{-24}$
9	1385	7520	$\sim 10^{-28}$
10	12661	31857	$\sim 10^{-31}$
11	87888	67382	$\sim 10^{-36}$

Table – Search results for small five-dimensional reflexive polytopes. The number of lattice points is listed in the first column. The total number of generated reflexive polytopes is given in the second column and the number of GA runs taken to reach this total is provided in the third column. The last column provides an upper bound on the fraction of states visiting during all GA runs.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GENETIC ALGORITHM SUCCESSFULLY GENERATES NEW 5D REFLEXIVE POLYTOPES

Formula for the Hodge number $h^{1,1}$ of a Calabi-Yau hypersurface X from a reflexive polytope Δ is given as :

$$h^{1,1}(X) = l(\Delta^*) - 6 - \sum_{\mathrm{codim}\Theta^*=1} l^*(\Theta^*) + \sum_{\mathrm{codim}\Theta=2} l^*(\Theta^*) \cdot l^*(\Theta)$$

where l is the number of points, l^* is the number of interior points, Δ is the dual polytope to Δ, Θ is a face of Δ and Θ^* is a face of Δ^*.

Conjecture

There are precisely 15 five-dimensional reflexive polytopes that give rise to four complex dimensional Calabi–Yau hypersurfaces with Hodge number $h^{1,1} = 1$.

Results : 5d targeted search

In eleven-dimensional supergravity compactified on CY fourfolds a condition necessary for unbroken $\mathcal{N}=1$ supersymmetry is that the Euler number χ of the CY fourfold must be divisible by $\delta \in \{24, 224, 504\}$.

To search for such cases, we modify our fitness function to be

$$\tilde{f}(\Delta) = w_1 \left(\mathrm{IP}(\Delta) - 1\right) - \frac{w_2}{k} \sum_{i=1}^{k} |a_i(\Delta) - 1| - w_3 \sum_{\delta} \chi(\Delta) \bmod \delta \, ,$$

where w_3 is a weight and $\chi(\Delta)$ is the Euler number of Δ.

Setting the `MIN` $= -3$, `BINLEN` $= 3$ and $m = 10$ and running the GA for 10 runs the GA finds 21 polytopes that satisfy the supersymmetry breaking condition.

Conclusions

- GA generated all 16 2d and all 4319 3d reflexive polytopes.
- GA generated all small (i.e. small number of points) 4d reflexive polytopes.
- Datasets of small 5d polytopes were generated using the GA.
- Generated 5d reflexive polytopes include new ones, leading to new Calabi–Yau fourfolds.
- Conjecture on the number of 5d reflexive polytopes giving rise to Calabi-Yau fourfolds with $h^{1,1} = 1$.
- Showcased the capability of a targeted search.

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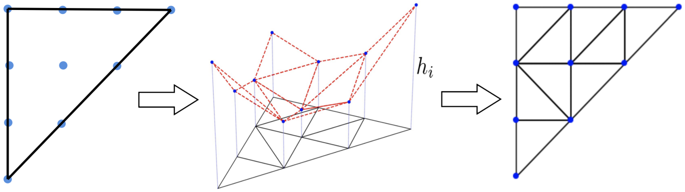
Outlook

A reflexive polytope in general produces a singular toric variety and therefore singular Calabi-Yau hypersurfaces. Resolving the singularities can be done by finding a fine, regular, star triangulation (FRST) of the reflexive polytope.



Can genetic algorithms generate triangulations? Could we combine the two genetic algorithms to generate phenomenologically viable Calabi-Yaus, e.g. with $h^{1,1} = 1$, $\chi = 6$ and K3 elliptically fibered.

Thank You

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