Machine Learning Sasakian and G2 topology on contact Calabi-Yau 7-manifolds

Dr Edward Hirst

Queen Mary, University of London
e.hirst@qmul.ac.uk

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...work with {Prof. David Berman, Prof. Yang-Hui He}; {Daattavya Aggarwal, Prof. Yang-Hui He, Elli Heyes, Prof. Henrique N. Sá Earp, Tomás S. R. Silva}

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Overview

1. Calabi-Yau Link Construction
   - $CY_3 \hookrightarrow \mathbb{P}^4_w$
   - CY Links
2. Data Analysis
   - Link Invariants
   - Link Conjectures
3. Supervised Learning
   - CY Topology
   - Link Topology
   - Gröbner Basis
4. Summary

Calabi-Yau 3-folds

Weighted Projective Spaces: $\mathbb{P}^4_w$

$\mathbb{C}^5 \longmapsto \mathbb{P}^4_w \longmapsto CY_3$
s.t. $(z_0, z_1, z_2, z_3, z_4) \sim (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2, \lambda^{w_3} z_3, \lambda^{w_4} z_4) \qquad \forall\, \lambda \in \mathbb{C}^*$
...then restrict to *anticanonical divisor* hypersurface.

For 7555 (w_i) vectors this hypersurface is a Calabi-Yau 3-fold.

Topological Invariant Formulas

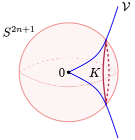
$$Q(u,v) = \sum_{p,q} h^{p,q} u^p v^q = \frac{1}{uv} \sum_{l=0}^{\sum_i (w_i)} \left[\prod_{\tilde{\theta}_i(l) \in \mathbb{Z}} \frac{(uv)^{q_i} - uv}{1 - (uv)^{q_i}} \right]_{int} \left(v^{size(l)} \left(\frac{u}{v}\right)^{age(l)} \right),$$

$$\chi = \frac{1}{\sum_i (w_i)} \sum_{l,r=0}^{\sum_i (w_i) - 1} \left[\prod_{i | lq_i \& rq_i \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) \right].$$

Calabi-Yau Links

Construction

Complex variety $\mathcal{V} \subset \mathbb{C}^{n+1}$, with isolated singularity at origin, transversally intersects S^{2n+1}_{ε}, defining a link: $K := \mathcal{V} \cap S^{2n+1}_{\varepsilon}$



Sasakian Structure (K, θ, g)

Where $\mathcal{V}$ a homogeneous polynomial f, link an S^1-bundle over this smooth projective hypersurface. S^1-bundle gives global angular contact form θ. Contact form gives Reeb vector ξ, and hence characteristic 1d foilation for transverse Kähler structure.

G2 Structure (K, φ)

Set $n = 4$ produces 7d CY link with cocalibrated G2 structure:

$$\varphi := \theta \wedge \omega + Im\Omega \ , \qquad \psi = *\varphi := \frac{1}{2}\omega \wedge \omega + \theta \wedge Re\Omega \ ,$$

...for 2-form $\omega = d\theta$, and CY_3 holomorphic volume (3,0)-form Ω.

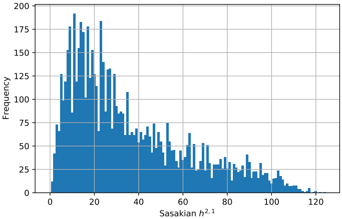
Link Topological Data

Sasakian $h^{p,q}$

Hodge numbers with $p+q=3$ given by dimension of linear subspaces of Milnor algebra $\mathbb{M}_f := \mathbb{C}[z_i]/(\partial f/\partial z_i)$:

$$h^{p,q}(K) = \dim_{\mathbb{C}}(\mathbb{M}_f)_\ell \ ,$$

for $\ell = (p+1)d - \sum_i w_i$. $h^{3,0}=1$, $h^{2,1} \rightarrow$

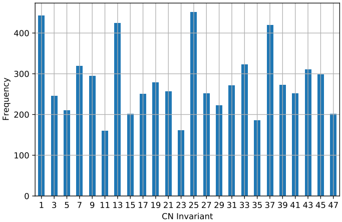


Crowley-Nordström ν

A $\mathbb{Z}_{48}$-valued homotopy invariant, computed from a compact coboundary Spin(7)-structure manifold (W_8, Ψ), such that $K = \partial W$ & $\Psi|_K = \varphi$:

$$\nu(\varphi) := \chi(W) - 3\sigma(W) \mod 48$$

for Euler characteristic χ and signature σ.



Link Conjectures

Weak R-Equivalence

Two weighted homogeneous polynomials with the same weights, but manifestly non-isomorphic Jacobi ideals, have the same dimensions of Milnor algebra linear subspaces.

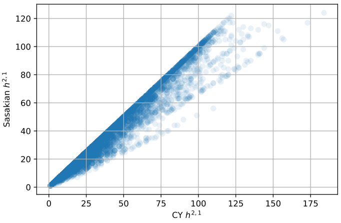
$\implies$ any homogeneous polynomial from the same weight system (w_i) has the same ν and Sasakian $h^{p,q}$ for $p+q=3$.

Link $h^{2,1}$ Restriction

The transverse Sasakian Hodge number $h_S^{2,1}$ for a Calabi-Yau link is bounded above by the equivalent Hodge number $h_{CY}^{2,1}$ of the Calabi-Yau manifold it was built from:

$$h_S^{2,1} \leq h_{CY}^{2,1} .$$

...also trivially applies for $h_S^{3,0} = h_{CY}^{3,0} = 1$.



Supervised Learning

Neural Network Architecture

Regressor Leaky-ReLU NN

Layers: (32,64,32), MSE loss, Adam, 5-fold cross-validation

Measure $R^2 = 1 - \frac{\sum(y_{true} - y_{pred})^2}{\sum(y_{true} - y_{truemean})^2} \in (-\infty, 1]$

$$(w_0, w_1, w_2, w_3, w_4) \longmapsto h^{p,q}$$

Calabi-Yau: $\{\chi, h^{1,1}, h^{2,1}\}$

Measure	Parameter		
	χ	$h^{1,1}$	$h^{2,1}$
R^2	0.9510 $\pm$ 0.0023	0.9630 $\pm$ 0.0015	0.9450 $\pm$ 0.0133

Sasakian: $\{h^{2,1}\}$

Measure	Parameter
	$h^{2,1}$
R^2	0.951 $\pm$ 0.012

Supervised Learning

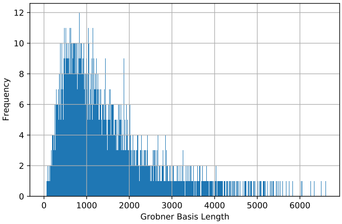
Gröbner Basis

By a significant margin, the computational bottleneck (in terms of both time and memory) of the invariant calculations was the generation of the *Gröbner basis*. Many initial runs failed from memory overload in this step for specific Calabi-Yau links. ML methods can predict the basis length to allow efficient HPC resource allocation.

Predicting Basis Length

$$(w_0, w_1, w_2, w_3, w_4) \longmapsto |GB|$$

Measure	Parameter $\lvert GB\rvert$
R^2	0.969 ± 0.002
MAE	107 ± 2



Summary

Summary Points

1) Generated the largest database of Calabi-Yau links.
2) Identified links with new ν-invariant values.
3) Conjectured the spectrum of topological invariants is *exhaustive* for $\mathbb{P}^4_w$ constructions.
4) Conjectured $h^{2,1}_S \leq h^{2,1}_{CY}$.
5) NNs can predict $h^{2,1}_S$ with high accuracies.
6) NNs can predict Gröbner basis length with high accuracies.

Outlook

1) Prove the raised conjectures.
2) Symbolically regress the $h^{2,1}_S$ NN formula.
3) Extend computation to other Hodge numbers.
4) Learn other Gröbner basis properties for more general ideals.

ML CY_3
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