Descendent series for Hilbert schemes of points

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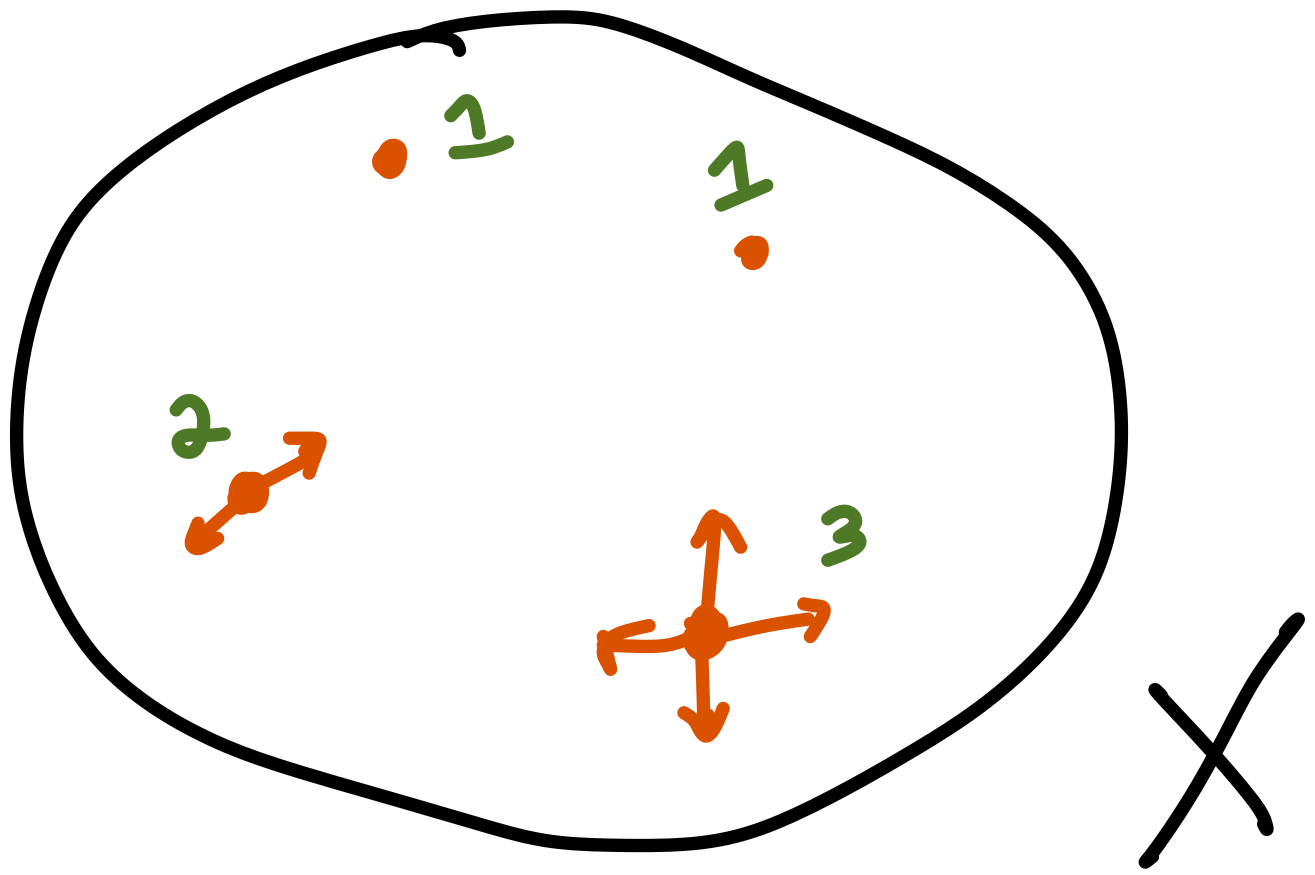
slides: math.columbia.edu/~nma/Nottingham.pdf

X – smooth quasiprojective variety $/\mathbb{C}$

$n \geq 0$

$$\rightsquigarrow X^{[n]} = \left\{ Z \subset X \;\middle|\; \begin{matrix} \dim(Z) = 0 \\ \dim_{\mathbb{C}} \Gamma(\mathcal{O}_Z) = n \end{matrix} \right\}$$ "Hilbert scheme of n points on X"

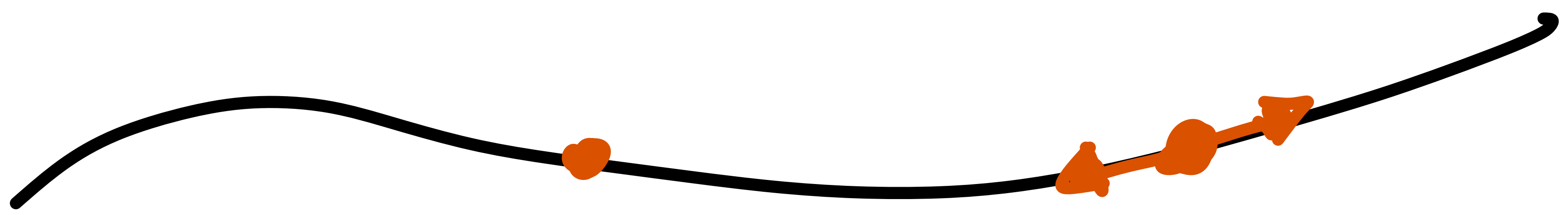


$$\in X^{[7]}$$

In general, $X^{[n]}$ are highly singular, of unknown dim^n

but are well-behaved when $\dim X \leq 2$.

C - a curve



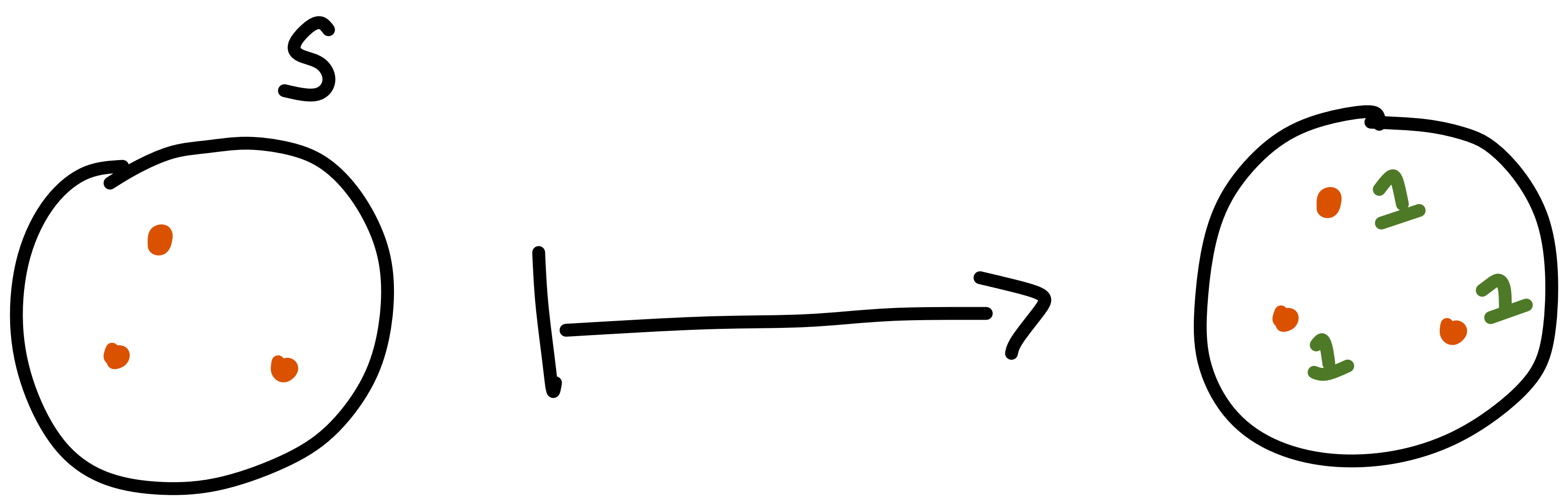
$$\left\{\begin{matrix}\text{length } n \text{ subschemes}\\ \text{of } C\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{effective divisors}\\ P_1+\dots+P_n\end{matrix}\right\}$$

$$C^{[n]} \cong \mathrm{Sym}^n C \longleftarrow \text{smooth}$$

eg/

$$\mathbb{P}^{1^{[n]}} \cong \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1}(n))) \cong \mathbb{P}^n$$

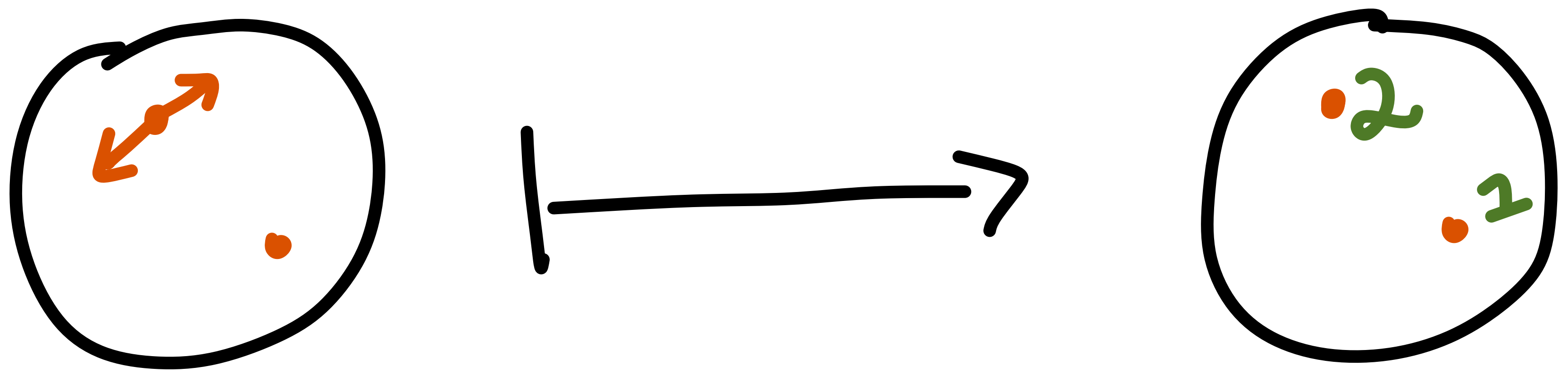
S - a surface



$$f : S^{[n]} \longrightarrow \mathrm{Sym}^n S$$

singular

$$Z \longmapsto \sum_{p \in \mathrm{Supp}\, Z} \mathrm{len}(\mathcal{O}_{Z,p}) \cdot p$$



Thm (Fogarty):

The Hilbert scheme $S^{[n]}$ is smooth of dimension $2n$.

The morphism

$$f : S^{[n]} \longrightarrow \mathrm{Sym}^n S$$

is a resolution of singularities.

Tautological bundles:

$$\begin{array}{ccl} V & \text{v.b. of rank } r & \\ \downarrow & & \\ X & & \end{array} \quad \rightsquigarrow \quad \begin{array}{c} V^{[n]} \\ \downarrow \\ X^{[n]} \end{array} \quad \begin{array}{l} \text{v.b. of rank } r \cdot n \text{ w/} \\ V^{[n]}\big|_{Z \in X^{[n]}} = \Gamma(V \otimes \mathcal{O}_Z) \end{array}$$

The bundles $V^{[n]}$ and their characteristic classes arise in geometric and physical computations.

Structure emerges when X is fixed but n can vary.

~~eg:~~ Fix: a projective surface S,

vector bundles $V_1, \ldots, V_\ell$ on S

integers $k_1, \ldots, k_\ell$

and consider the power series:

$$\sum_{n \geq 0} q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right)$$

"K-theoretic" descendent series

(Curves:) Example 2:

$$\sum_{n\geq 0} q^n \chi(\mathbb{P}^{1[n]}, \mathcal{O}_{\mathbb{P}^{1[n]}}) = \sum_{n\geq 0} q^n \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$$

$$= \sum_{n\geq 0} q^n$$

$$= \frac{1}{1-q} \left(= \frac{1}{(1-q)^{\chi(\mathcal{O}_{\mathbb{P}^1})}} \right)$$

(Curves:) Example-1: If C is a curve:

$$\sum_{n\geq 0} q^n \chi\left(C^{[n]}, \mathcal{O}_{C^{[n]}}\right) = \sum_{n\geq 0} q^n \chi\left(\mathrm{Sym}^n C, \mathcal{O}_{\mathrm{Sym}^n C}\right)$$

$$= \sum_{n\geq 0} q^n \, g.\dim H^\bullet\left(C^n, \mathcal{O}_{C^n}\right)^{G_n}$$

$$= \sum_{n\geq 0} q^n \, g.\dim \left(H^\bullet(C, \mathcal{O}_C)^{\otimes n}\right)^{G_n}$$

$$= \frac{(1-q)^{h^1(C,\mathcal{O}_C)}}{(1-q)^{h^0(C,\mathcal{O}_C)}} = \boxed{\frac{1}{(1-q)^{\chi(\mathcal{O}_C)}}}$$

Surfaces: Example 0: If S is a surface:

$$\sum_{n\geq 0} q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = ?$$

$$f : S^{[n]} \longrightarrow \mathrm{Sym}^n S$$

$\mathrm{Sym}^n S$ has rational singularities $\longrightarrow R^i f_* \mathcal{O}_{S^{[n]}} = 0, \ i>0.$

f proper, birational w/ normal target $\longrightarrow R^0 f_* \mathcal{O}_{S^{[n]}} = \mathcal{O}_{\mathrm{Sym}^n S}$

$$\sum_{n\geq 0} q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = \sum_{n\geq 0} q^n \chi(\mathrm{Sym}^n S, \mathcal{O}_{\mathrm{Sym}^n S}) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}}$$

Thm [A]: Fix: a projective surface S,

vector bundles $V_1, \ldots, V_\ell$ on S

integers $k_1, \ldots, k_\ell \geq 0$.

Then,

$$\sum_{n \geq 0} q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right) = \frac{\text{Polynomial in } q \text{ of degree} \leq k_1 + \cdots + k_\ell}{(1-q)^{\chi(\mathcal{O}_S)}}.$$

Remarks:

$$\sum_{n\geq 0} q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right)$$

$$\parallel$$

$$\frac{\text{polynomial in } q \text{ of degree} \leq k_1 + \ldots + k_\ell}{(1-q)^{\chi(\mathcal{O}_S)}}$$

- $1/\text{DENOMINATOR} = \sum_{n\geq 0} \chi\left(S^{[n]}, \mathcal{O}_{S^{[n]}}\right)$
- Special cases (small ℓ, k_ℓ, $\operatorname{rk} V_j$) computed by

  [Danila, Krug, Scala, Zhou ...]
- Analogous statement holds when S is replaced by a curve C

  [Oprea - Pandharipande]

Example 1: If V is a rank 2 vector bundle on S, then

$$\sum_{n\geq 0} q^n \chi(S^{[n]}, \Lambda^3 V^{[n]})$$

$$\|$$

$$\frac{q^3\binom{\chi(V)}{3} + (q^2 - q^3)\left(\chi(V)\chi(\Lambda^2 V) - \chi(V\otimes\Lambda^2 V)\right)}{(1-q)^{\chi(\mathcal{O}_S)}}$$

eg.

Note: coefficients are polynomials in Euler characteristics of V and TS (and Schur functors thereof)

[Ellingsrud-Göttsche-Lehn]

"cohomological descendent series":

Fix surface S, v. bundles $V_1, \dots, V_\ell$, integers $k_1, \dots, k_\ell \geq 0$.

Form $$\sum_{n \geq 0} q^n \int_{S^{[n]}} \mathrm{ch}_{k_1}(V_1^{[n]}) \cdots \mathrm{ch}_{k_\ell}(V_\ell^{[n]}) \, c_{tot}(TS^{[n]})$$

These behave differently from their K-theoretic analogues.

e.g. $$\sum_{n \geq 0} q^n \int_{S^{[n]}} e(S^{[n]}) = \prod_{r > 0} \frac{1}{(1-q^r)^{e(S)}}$$

not rational!

e.g. [Carlsson - Okounkov]

$$\sum_{n\geq 0} q^n \int_{S^{[n]}} c_1(\mathcal{O}^{[n]})\, c_{2n-1}(TS^{[n]})$$

$$=$$

$$\frac{\frac{c_1(S)^2}{2} \cdot \left(\sum_{m>0} \frac{(m-m^2)\, q^m}{1-q^m} \right)}{\prod_{r>0} (1-q^r)^{e(S)}}$$

q-deformation of $\zeta(2) - \zeta(3)$

Conjecture [Okounkov]:

$$\sum_{n\geq 0} q^n \int_{S^{[n]}} \mathrm{ch}_{k_1}(V_1^{[n]}) \cdots \mathrm{ch}_{k_\ell}(V_\ell^{[n]})\, c_{tot}(TS^{[n]})$$

belongs to a ring of "q-multiple zeta values"

$$\left(\text{particular } q\text{-deformations of } \sum_{m_1 > \cdots > m_j} \frac{1}{m_1^{s_1} \cdots m_j^{s_j}}\right)$$

In contrast, if C is a curve, $W_1, \ldots, W_\ell$ are vector bundles on C (and $k_1, \ldots, k_\ell \geq 0$), then:

Thm: [Johnson-Oprea-Pandharipande]

$$\sum_{n \geq 0} q^n \int_{C^{[n]}} ch_{k_1}(W_1^{[n]}) \cdots ch_{k_\ell}(W_\ell^{[n]}) \, c_{tot}(TC^{[n]})$$

is the Laurent expansion of a rational function in q.

flavor of descendent series \ moduli space	Hilbert scheme on curve	Hilbert scheme on surface
cohomological $\sum_n q^n \int_{X^{[n]}} \cdots$	rational function in q	q-multiple zeta value (conjecturally)
K-theoretic $\sum_n q^n \chi(X^{[n]}, \ldots)$	rational function in q	rational function in q

Outline of proof that:

$$\sum_{n\geq 0} q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right) = \frac{\text{polynomial in } q \text{ of degree} \leq k_1 + \ldots + k_\ell}{(1-q)^{\chi(\mathcal{O}_S)}}$$

for all $S, V_1, \ldots, V_\ell, k_1, \ldots, k_\ell$.

Step 1: Reduce to case where S is a toric surface.

We saw that for $\mathcal{V}$ of rank 2:

$$\chi(S^{[2]}, \Lambda^3 \mathcal{V}^{[2]}) = \chi(S,\mathcal{V}) \cdot \chi(S, \Lambda^2 \mathcal{V}) - \chi(S, \mathcal{V} \otimes \Lambda^2 \mathcal{V})$$

In general, a result of [Ellingsrud-Göttsche-Lehn] implies that there exists a "universal polynomial" $P_{k,r,n} \in \mathbb{Q}[x_1, \ldots, x_5]$ such that for all S and all $\mathcal{V}$ of rank r, one has

$$\chi(S^{[n]}, \Lambda^k \mathcal{V}^{[n]}) = P_{k,r,n}\left(c_1(S)^2, c_2(S), c_1(\mathcal{V})^2, c_2(\mathcal{V}), c_1(S)c_1(\mathcal{V})\right)$$

For all S and all V of rank r, one has

$$\chi(S^{[n]}, \Lambda^k V^{[n]})$$
$$\| $$
$$P_{k,r,n}\left(c_1(S)^2, c_2(S), c_1(V)^2, c_2(V), c_1(S)c_1(V)\right).$$

The polynomial $P_{k,r,n}$ uniquely determined by computing the value of $\chi(S_i^{[n]}, \Lambda^k V_i^{[n]})$ for sufficiently many $\{(S_i, V_i)\}$ such that $\{(c_1(S_i)^2, c_2(S_i), c_1(V_i)^2, c_2(V_i), c_1(S_i)c_1(V_i))\}_{i=1}^{s}$ separate monomials

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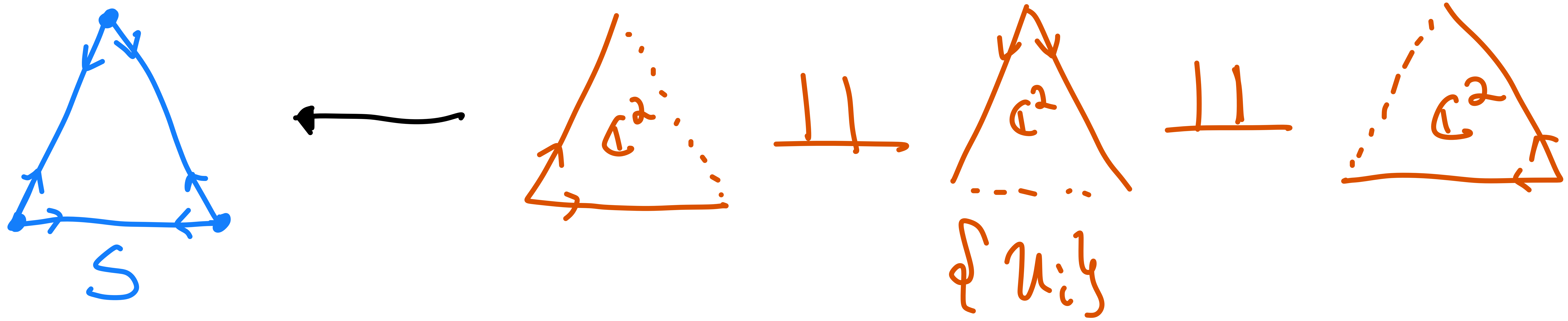
In particular, suffices to compute for toric S_i (and V_i torus-equivariant.)

Step 2: Reduce from toric S to (equivariant version with) $S = \mathbb{C}^2$.

Let $T \circlearrowright S$ with toric charts $\{\mathcal{U}_i\}$. Equivariant localization [Thomason] implies that the restriction map

$$K_{T,\mathrm{loc}}\Big(\coprod_n S^{[n]}\Big) \longrightarrow K_{T,\mathrm{loc}}\Big(\coprod_i \coprod_{n_i} \mathcal{U}_i^{[n_i]}\Big)$$

is an isomorphism.



Step 2: Reduce from toric S to (equivariant version with) $S=\mathbb{C}^2$.

Convenient to introduce a new variable y and form:

$$\phi(S,V) = \sum_{n,k} q^n y^k \chi(S^{[n]}, \Lambda^k V^{[n]})$$

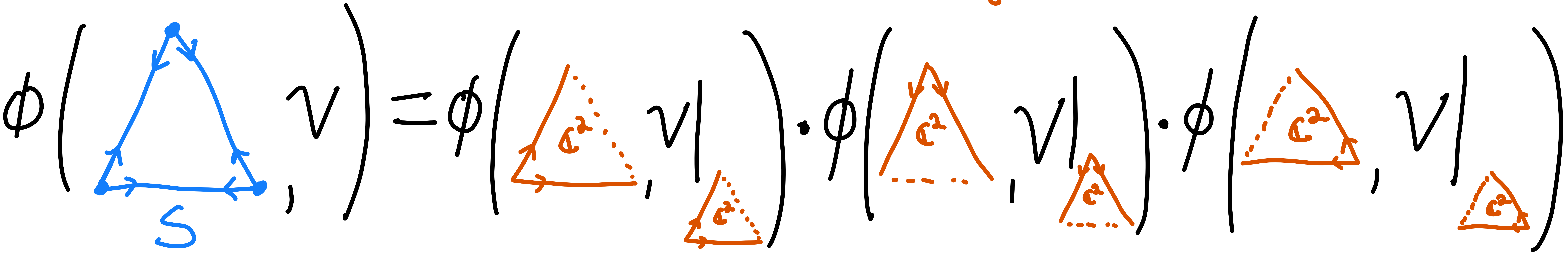
Advantage: If $S = S' \sqcup S''$, then

$$\phi(S,V) = \phi(S', V|_{S'}) \cdot \phi(S'', V|_{S''})$$

$$\boxed{\phi(S,V) = \sum_{n,k} q^n y^k \chi\left(S^{[n]}, \Lambda^k V^{[n]}\right)}$$

In particular, if S is toric with toric charts $\{U_i\}$:

$$\phi(S,V) = \prod_i \phi\Big(\underset{\underset{\mathbb{C}^2_i}{\wr}}{U_i}, V|_{U_i}\Big).$$



Conclusion: ϕ can be reconstructed for toric S from its values for $\mathbb{C}^2$.

Note: $\mathbb{C}^{2[n]}$ is not proper! So, $\chi(\mathbb{C}^{2[n]}, \Lambda^k V^{[n]})$ is regarded as a T-equivariant Euler characteristic.

If $T = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$, then $\chi(\mathbb{C}^{2[n]}, \Lambda^k V^{[n]}) \in \mathbb{Q}(t_1, t_2)$.

(e.g. if coordinates on $\mathbb{C}^2$ are scaled by t_1, t_2, then $\chi(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) = \dfrac{1}{(1-t_1)(1-t_2)}$)

$$\phi(S, V) = \prod_i \phi(\underset{\substack{\cong \\ \mathbb{C}^2_i}}{\mathcal{U}_i}, V|_{\mathcal{U}_i}).$$

read as an equality of Laurent polynomials in t_1, t_2.

(non-equivariant Euler characteristics $\phi(S,V)$ obtained by specializing $t_1 = t_2 = 1$)

Thm reduces to following proposition: $T = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \curvearrowright \mathbb{C}^2$

If $W_1, \ldots, W_\ell$ are T-equivariant bundles on $\mathbb{C}^2$, and $k_1, \ldots, k_\ell \geq 0$.

Then

$$\sum_{n \geq 0} q^n \chi\left(\mathbb{C}^{2[n]}, \Lambda^{k_1} W_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} W_\ell^{[n]}\right)$$

$$\Vert$$

$$\frac{\text{polynomial in } q \text{ of deg} \leq k_1 + \cdots + k_\ell \text{ (w/ coefficients in } \mathbb{Q}(t_1, t_2))}{\prod_{i_1, i_2 \geq 0} 1 - q t_1^{i_1} t_2^{i_2}}$$

$$\sum_{n\geq 0} q^n \chi\left(\mathbb{C}^{2^{[n]}}, \Lambda^{k_1} W_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} W_\ell^{[n]}\right)$$

studied by:

1) using equivariant localization on $\mathbb{C}^{2^{[n]}}$ to write series combinatorially (as a sum over Young diagrams.)

2) controlling the resulting combinatorial expression using a Macdonald polynomial identity of Mellit.

(More) localization [eg. Thomason]:

If $T \curvearrowright \underset{M}{\text{smooth}}$ w/ isolated fixed points and $\mathcal{F} \in K_T(M)$

then

$$\chi(M, \mathcal{F}) = \sum_{p \in M^T} \chi\left(p, \frac{\mathcal{F}|_p}{\sum_i \Lambda^i T_p^* M}\right)$$

Mnemonic: Suppose M has Čech cover of open balls $\{\mathcal{U}_p\}$ centered at $p \in M^T$. Then p-term of RHS is $\chi(\mathcal{U}_p, \mathcal{F}|_{\mathcal{U}_p})$.

Applied to

$$\sum_{n,k} q^n y^k \chi\left(\mathbb{C}^{2[n]}, \Lambda^k W^{[n]}\right)$$

$$\boxed{\chi(M,F) = \sum_{p \in M^T} \chi\left(p, \frac{F|_p}{\sum_i \Lambda^i T_p^* M}\right)}$$

$\left(\mathbb{C}^{2[n]}\right)^T \longleftrightarrow \left\{ \begin{array}{l} \text{monomial ideals} \\ I_\lambda \subset \mathbb{C}[x_1, x_2] \\ \text{of colength } n \end{array} \right\} \longleftrightarrow$ Young diagrams λ of size n

1	x_2	x_2^2	x_2^3	$\cdots$
x_1	$x_1 x_2$	$x_1 x_2^2$	$x_1 x_2^3$	$\cdots$
x_1^2	$x_1^2 x_2$	$x_1^2 x_2^2$	$x_1^2 x_2^3$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Applied to $\sum_{n,k} q^n y^k \chi\left(\mathbb{C}^{2[n]}, \Lambda^k W^{[n]}\right)$

$\left(\mathbb{C}^{2[n]}\right)^T \longleftrightarrow$ Young diagrams λ of size n

$$\boxed{\chi(M,\mathcal{F}) = \sum_{p \in M^T} \chi\left(p, \frac{\mathcal{F}|_p}{\sum_i \Lambda^i T_p^* M}\right)}$$

- if $\chi(W|_{0\in\mathbb{C}^2}) = \sum_j w_j$, then

$$\sum_k y^k \chi\left(\lambda, \Lambda^k W^{[n]}|_\lambda\right) =: \prod_j N_\lambda(y \cdot w_j)$$

(N_λ is some explicit comb. expression of degree $|\lambda|$ in terms of monomials appearing in λ)

- $\chi\left(\lambda, \frac{1}{\sum_i \Lambda^i T_\lambda^* \mathbb{C}^{2[n]}}\right) =: D_\lambda$, some explicit combinatorial expression in terms of "arm" + "leg" lengths of λ.

So $$\sum_{n,k} q^n y^k \left(\mathbb{C}^{2[n]}, \Lambda^k W^{[n]}\right) = \sum_{\lambda} q^{|\lambda|} \frac{\prod_j N_\lambda(y \cdot w_j)}{D_\lambda}$$

Let $\{H_\lambda\}$ be the Macdonald polynomials (certain homogenous polynomials of degree $|\lambda|$)

Then, an identity of [Mellit] implies the following duality:

$$\frac{\sum_{\lambda} q^{|\lambda|} \frac{\prod_j N_\lambda(y w_j)}{D_\lambda}}{\prod_{\substack{i_1, i_2 \\ j}} \left(1 - y w_j t_1^{i_1} t_2^{i_2}\right)} = \frac{\sum_{\lambda} H_\lambda(y \cdot w_j) \frac{N_\lambda(q)}{D_\lambda}}{\prod_{i_1, i_2} \left(1 - q t_1^{i_1} t_2^{i_2}\right)}$$

Conclusion:

$$\sum_{n,k} q^n y^k \left(\mathbb{C}^2[n], \Lambda^k W^{[n]}\right)$$

$$\|$$

$$\sum_{\lambda} q^{|\lambda|} \frac{\prod_j N_\lambda(y \cdot w_j)}{D_\lambda}$$

$$\|$$

$$\frac{\prod_{i_1,i_2,j}\left(1 - y w_j t_1^{i_1} t_2^{i_2}\right)}{\prod_{i_1,i_2}\left(1 - q t_1^{i_1} t_2^{i_2}\right)} \cdot \sum_\lambda H_\lambda(y \cdot w_j) \frac{N_\lambda(q)}{D_\lambda}$$

eg y^3-term:

all λ contribute here

but

only λ of size ≤ 3 contribute here

Desired polynomiality follows!

Hence,

$$\sum_{n\geq 0} q^n \chi\left(\mathbb{C}^{2[n]}, \Lambda^{k_1} W_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} W_\ell^{[n]}\right) = \frac{\text{polynomial in } q \text{ of deg} \leq k_1 + \ldots + k_\ell \text{ (w/ coefficients in } \mathbb{Q}(t_1, t_2))}{\prod_{i_1, i_2 \geq 0} 1 - q t_1^{i_1} t_2^{i_2}}$$

so that

$$\sum_{n\geq 0} q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right) = \frac{\text{polynomial in } q \text{ of degree} \leq k_1 + \ldots + k_\ell}{(1-q)^{\chi(\mathcal{O}_S)}}.$$

Thank you!