Gorenstein Fano 3-folds of Picard number 1 with a 2-torus action

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github.com/abaeuerle/fano-3d-lt-gor-rho-1

$\mathbb{Q}$-factorial Fano 3-folds with a torus action

$T \curvearrowright X$ 3-dim. norm. $\mathbb{Q}$-fact. proj. variety $/\mathbb{C}$, $-K_X$ ample, log terminal.

Complexity: $\dim X - \dim T$, Picard number: $\rho = \mathrm{rk}(\mathrm{Pic}(X))$, Gorenstein index: $\iota = \min\{k \geq 1 ; -kK_X \text{ Cartier}\}$

Some classification results:

	toric		cplx 1			cplx ≥ 2		
smooth:	18	[Ba81]		20	[Süß10]		67	[Is77/78], [MoMu81/82]
terminal:	233 (634)	[Ka06]	($\rho = 1$)	47	[BeHaHuNi16]			
canonical:	12,190 (674,688)	[Ka10]	($\rho = 1$, $\iota = 1$)	538	[_Ha21]	(cplx 2)	≥ 49	[HiWr19]

[BrKa22]: $\leq 39{,}550$ Hilbert series of sc. Mori-Fano 3-folds

Fake weighted projective spaces

$$H = \mathbb{C}^* \times C, \quad C = C(m_1) \times \dots \times C(m_k)$$

$$\mathbb{X}(H) \cong \mathbb{Z} \times \Gamma, \quad \Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_k\mathbb{Z}$$

$$w_0, \dots, w_d \in \mathbb{Z} \times \Gamma, \quad w_i = (w_i, \eta_i), \quad w_0, \dots, w_d > 0$$

$$H \curvearrowright \mathbb{C}^{d+1}, \quad h \cdot z = (\chi^{w_0}(h) z_0, \dots, \chi^{w_d}(h) z_d)$$

$$Z = \mathbb{P}(w_0, \dots, w_d) := (\mathbb{C}^{d+1} \setminus \{0\}) / H$$

Hom. coord. $[z_0, \dots, z_d]$, $[z] = [z'] \iff z' = h \cdot z$

$$\mathbb{C}[T_0, \dots, T_d] = \bigoplus_{w \in \mathbb{X}(H)} \mathbb{C}[T_0, \dots, T_d]_w$$

$$D_i = V(T_i) := \{[z] \in Z;\ z_i = 0\}, \quad U_i := Z \setminus D_i$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{PDiv}^T(Z) \arrow[r] \arrow[d, "\cong"', "\operatorname{div}(\chi^u) \mapsto \chi^u"] & \mathrm{WDiv}^T(Z) \arrow[r] \arrow[d, "\cong"', "\sum a_i D_i \mapsto \chi^a"] & \mathrm{Cl}(Z) \arrow[r] \arrow[d, "\cong"', "{[D_i] \mapsto \chi^{w_i}}"] & 0 \\
0 \arrow[r] & \mathbb{X}(T) \arrow[r] & \mathbb{X}(\mathbb{T}^{d+1}) \arrow[r] & \mathbb{X}(H) \arrow[r] & 0
\end{tikzcd}$$

$$H = \mathbb{C}^* \times \{\pm 1\}$$

$$\mathbb{X}(H) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$(w_0, \dots, w_4) = \begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix}$$

$$(t, \zeta) \cdot z = (t^2 \zeta z_0, t^2 z_1, t^2 z_2, t^3 \zeta z_3, t z_4)$$

$$\mathbb{P}\begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix} = (\mathbb{C}^5 \setminus \{0\}) / \mathbb{C}^* \times \{\pm 1\}$$

$$f = T_0^2 T_1 + T_2^3 + T_3^2, \quad \deg(f) = (6, \bar{0}) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$U_4 \cong \operatorname{Spec}(\mathbb{C}[T_1, \dots, T_5] / \langle T_1^2 - T_2 T_3 \rangle)$$

A natural embedding

X non-toric $\mathbb{Q}$-fact. Fano 3-fold, cplx 1, $\rho = 1$.

There are (i) $r \geq 2$, (ii) $n_0 \geq \ldots \geq n_r \geq 1$, (iii) $m \geq 0$,

(iv) $l_i \in \mathbb{Z}^{n_i}_{\geq 1}$, $n_i l_{i1} > 1$, (v) $\lambda_1, \ldots, \lambda_{r-2} \in \mathbb{C}^* \setminus \{1\}$,

with $n + m = n_0 + \ldots + n_r + m = r + 3$, such that:

$$X = V\begin{pmatrix} T^{l_0} + T^{l_1} + T^{l_2} & & \\ \lambda_1 T^{l_1} + T^{l_2} + T^{l_3} & & \\ \ddots & & \\ \lambda_{r-2} T^{l_{r-2}} + T^{l_{r-1}} + T^{l_r} & & \end{pmatrix} \subseteq \mathbb{P}(\omega_{ij}, \omega_k;\ 0 \leq i \leq r,\ 1 \leq j \leq n_i,\ 1 \leq k \leq m) = Z$$

with $\deg(g_0) = \ldots = \deg(g_{r-2})$, $D_{ij}|_X$, $D_k|_X$ pw. dist. prime div., $Cl(X) \cong Cl(Z)$.

Adjunction formula: $-K_X = -K_Z|_X - (r-1)\deg(g_0)$.

$(r; n_0, \ldots, n_r; m)$, l_i, w_{ij}, w_k unique!

Goal: Good bounds.

$r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 1$,

$l_0 = (2, 1)$, $l_1 = (3)$, $l_2 = (2)$.

$$X = V(T_{01}^2 T_{02} + T_{11}^3 + T_{21}^2)$$

$$\subseteq \mathbb{P}\begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix}$$

$$\deg(g_0) = \begin{pmatrix} 6 \\ \bar{0} \end{pmatrix}$$

$$-K_X = \begin{pmatrix} 10 \\ \bar{0} \end{pmatrix} - \begin{pmatrix} 6 \\ \bar{0} \end{pmatrix} = \begin{pmatrix} 4 \\ \bar{0} \end{pmatrix} \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Log terminality

Theorem [BeHaHuN:16]:

X $\mathbb{Q}$-fact. cplx 1 Fano 3-fold, $\rho = 1$. Equiv. are:

(i) X log terminal.

(ii) Every $(l_{0j_0}, \dots, l_{rj_r})$ up to order $(a,b,c,1,\dots,1)$, where

$(a,b,c) = (x,y,1),\ (y,2,2),\ (5,3,2),\ (4,3,2),\ (3,3,2).$

In this case: $r \leq 5$.

Explicitly: 8 cases for format $(r; n_0, \dots, n_r; m)$.

$X = V(T_{01}^2 T_{02} + T_{11}^3 + T_{21}^2)$

$((l_{01}, l_{11}, l_{21}) = (2,3,2)$. Case $(a,b,c) = (y,2,2)$

$((l_{02}, l_{11}, l_{21}) = (1,3,2)$. Case $(a,b,c) = (x,y,1)$

(i) $r = 2$, $n_0 = n_1 = 2$, $n_2 = 1$, $m = 0$.

(ii) $r = 3$, $n_0 = n_1 = 2$, $n_2 = n_3 = 1$, $m = 0$.

(iii) $r = 4$, $n_0 = n_1 = 2$, $n_2 = n_3 = n_4 = 1$, $m = 0$.

(iv) $r = 2$, $n_0 = 3$, $n_1 = n_2 = 1$, $m = 0$.

(v) $r = 3$, $n_0 = 3$, $n_1 = n_2 = n_3 = 1$, $m = 0$.

(vi) $r = 2$, $n_0 = 2$, $n_1 = n_2 = 1$, $m = 1$.

(vii) $r = 3$, $n_0 = 2$, $n_1 = n_2 = n_3 = 1$, $m = 1$.

(viii) $r = 2$, $n_0 = n_1 = n_2 = 1$, $m = 2$.

Gorenstein

$X \subseteq Z$. $Cl(X) \cong Cl(Z)$. In general $Pic(X) \neq Pic(Z)$.

$Z' \underset{open}{\subseteq} Z$ min. with $X \subseteq Z'$. $Pic(X) = Pic(Z')$.

$v_0, \dots, v_d$ prim. ray gen. of $\Sigma(Z')$, $U_i \subseteq Z'$.

$D = a_0 D_0 + \dots + a_d D_d$ Cartier $\quad D|_{U_i} = div(\chi^u)|_{U_i}$

$D - div(\chi^u) = (a_i - \langle u, v_i \rangle) D_i$

$X = V(T_0^2 T_1 + T_2^y + T_3^2) \subseteq Z' \subseteq \mathbb{P}(\omega_0, \dots \omega_4)$, $\omega_i = (w_i, \eta_i)$

$-k_X = -w_0 + w_2 + w_3 + w_4 = a_i w_i$, $i = 0, 1, 4$

$$\langle (w_0, \dots, w_4) \rangle = \ker \begin{pmatrix} -1-a_0 & 0 & 1 & 1 & 1 \\ -1 & -a_1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1-a_4 \\ -2 & -1 & y & 0 & 0 \\ -2 & -1 & 0 & 2 & 0 \end{pmatrix} =: G$$

$$1 = \frac{1}{a_4} + \frac{1}{2a_1} + \frac{1}{y}\left(\frac{2}{a_0} + \frac{1}{a_1}\right) = \frac{1}{a_0 y} + \frac{1}{a_0 y} + \frac{1}{2a_1} + \frac{1}{a_1 y} + \frac{1}{a_4}$$

$X = V(T_0^2 T_1 + T_2^3 + T_3^2) \subseteq \mathbb{P}\begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix} = Z$

$Cl(X) = Cl(Z) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$Pic(X) = \langle \binom{4}{\bar{0}} \rangle \supsetneq \langle \binom{12}{\bar{0}} \rangle = Pic(Z)$

Max. cones of Z':

cone(v_1, v_2, v_3, v_4), cone(v_0, v_2, v_3, v_4),
cone(v_0, v_1, v_2, v_3), cone(v_0, v_1, v_4).

$y = 3$, $(w_0, \dots, w_4) = (2, 2, 2, 3, 1)$

$-k_X = 4 = 2 \cdot w_0 = 2 \cdot w_1 = 4 \cdot w_4$,

$a_0 = 2$, $a_1 = 2$, $a_4 = 4$.

$$G = \begin{pmatrix} -3 & 0 & 1 & 1 & 1 \\ -1 & -2 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & -3 \\ -2 & -1 & 3 & 0 & 0 \\ -2 & -1 & 0 & 2 & 0 \end{pmatrix}$$

$$1 = \frac{1}{4} + \frac{1}{2 \cdot 2} + \frac{1}{3}\left(\frac{2}{2} + \frac{1}{2}\right) = \frac{1}{6} + \frac{1}{6} + \frac{1}{4} + \frac{1}{6} + \frac{1}{4}$$

Classification

X $\mathbb{Q}$-fact. log term. Gorenstein Fano 3-fold, $\rho = 1$, cplx 1.

log terminal

8 formats $(r, n_0, \dots, n_r, m)$

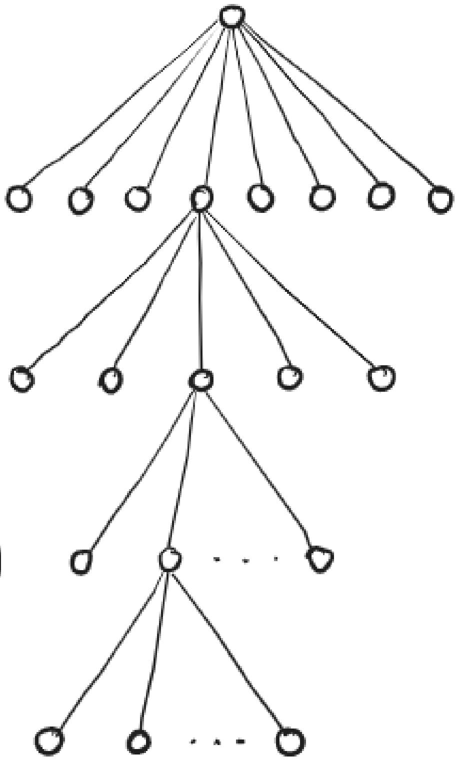
log terminal

5 plat. exp.-conf.

Gorenstein

Matrices G / weights $(w_0, \dots, w_d)$

Embedding $X \subseteq \mathbb{P}(w_0, \dots, w_d)$



$$(r, n_0, \dots, n_r, m) = (2, 2, 1, 1, 1)$$

$$(l_{01}, l_{11}, l_{21}) = (2, 3, 2) \; ; \; (y, 2, 2)$$

$$(l_{02}, l_{11}, l_{21}) = (1, 3, 2) \; ; \; (x, y, 1)$$

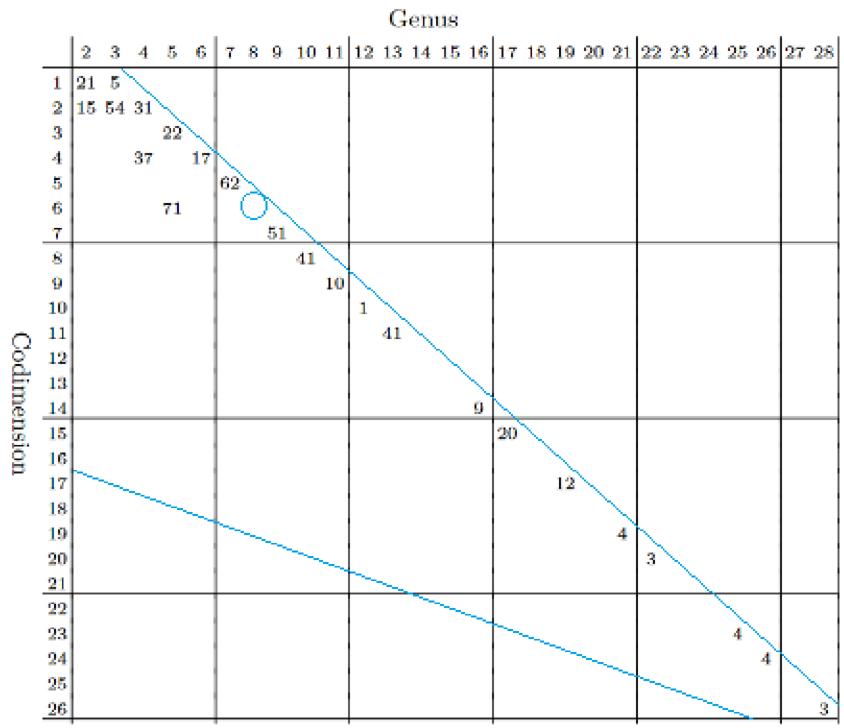
$$a_0 = a_1 = 2, \quad a_4 = 4, \quad (w_1, \dots, w_4) = (2, 2, 2, 3, 1)$$

$$V(T_0^2 T_1 + T_2^3 + T_3^2) \subseteq \begin{cases} \mathbb{P}(2, 2, 2, 3, 1) \\ \mathbb{P}\begin{pmatrix} 2 & 2 & 2 & 3 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix} \end{cases}$$

Classification results

Theorem [_Ha]: There are 538 families of non-toric, $\mathbb{Q}$-factorial, log terminal, Gorenstein Fano 3-folds of complexity 1 and Picard number 1. Every such X is isomorphic to exactly one member of one of those families.

divisor class group	*sporadic varieties*	*true families*
$\mathbb{Z}$	*242*	*3 one-dim.*
$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	*163*	*4 one-dim.*
$\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^2$	*46*	*5 one-dim.,* *1 two-dim.*
$\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^3$	*6*	*1 one-dim.*
$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	*4*	*1 one-dim.*
$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	*1*	*0*
$\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	*26*	*1 one-dim.*
$\mathbb{Z}\times(\mathbb{Z}/3\mathbb{Z})^2$	*1*	*0*
$\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	*18*	*1 one-dim.*
$\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$	*4*	*0*
$\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	*8*	*0*
$\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$	*2*	*0*



$g := h^0(X, -K_X) - 2, \quad (-K_X)^3 = 2g-2$

Outlook

Direct generalization for $c \geq 2$.

Direct generalization to higher dimensions

Almost* every smooth Fano 3-fold with $\rho = 1$ has a small one-parameter degeneration to a normal Gorenstein Fano 3-fold of cplx ≤ 1.

(* Constructed explicitly except for X_{14})

Thank you for your attention!