Orientations for DT invariants on quasi-projective Calabi-Yau 4-folds

Arkadij Bojko

University of Oxford

How to count coherent sheaves on CY4?

- Solutions to this have been given independently by Borisov–Joyce(15') and Cao–Leung(for some cases), algebraic geometric construction of Oh–Thomas(20').
- One considers three terms instead: $\mathrm{Ext}^1(E,E)$, $\mathrm{Ext}^2(E,E)$ and $\mathrm{Ext}^3(E,E)$. Using Serre duality, one has $\mathrm{Ext}^1(E,E) \cong (\mathrm{Ext}^3(E,E))^*$ and a non-degenerate billinear form $q : \mathrm{Ext}^2(E,E) \times \mathrm{Ext}^2(E,E) \to \mathbb{C}$.
- Slogan: "Take half of everything". Taking $\mathrm{Ext}^1(E,E)$ we covered $\mathrm{Ext}^3(E,E)$. Equivalently one can take a real subspace in $V = \mathrm{Ext}^1(E,E) \oplus \mathrm{Ext}^3(E,E)$ invariant under the isomorphism above together with $(\mathrm{Ext}^3(E,E))^* \cong \overline{\mathrm{Ext}^3(E,E)}$. Then $V_{\mathbb{R}}$ has a natural orientation compatible with the choice $\mathrm{Ext}^1(E,E)$ as an *isotropic subspace*.
- Problem: How to choose an isotropic subspace of $\mathrm{Ext}^2(E,E)$? Equivalently, how to choose orientation on $\mathrm{Ext}^2(E,E)_{\mathbb{R}}$? Can this choice be fit together continuously?

- Reformulate: Let $V_{\mathbb{R}} \subset V = (\mathrm{Ext}^2(E, E))$ be the real subspace for a real structure $\nu : V \to V^* \cong \bar{V}$. Then an orientation on $V_{\mathbb{R}}$ is equivalent to an orientation on $\det_{\mathbb{R}}(V_{\mathbb{R}}) \subset \det(V)$, which is a choice of isomorphism $\det_{\mathbb{R}}(V_{\mathbb{R}}) \cong \mathbb{R}$.
- In other words, we are looking for $o : \det(V) \to \mathbb{C}$, such that

$$o \otimes o = (\det(V)^{\otimes 2} \xrightarrow{\mathrm{id} \otimes \nu} \det(V)\det^*(V) \to \mathbb{C})\,.$$

Definition

$L \to S$ a complex line bundle over S, $\mu : L \to L^*$ an isomorphism, then one can define *square root* $\mathbb{Z}_2$*-bundle associated with* μ denoted by O^μ. This bundle is given by the sheaf of its sections:

$$O^\mu(U) = \{o : L|_U \xrightarrow{\sim} \underline{\mathbb{C}}_U : o \otimes o = \mathrm{ad}(\mu)|_U\}\,.$$

Baby version of theorem

- Let X be a quasi-projective Calabi–Yau 4-fold and M a quasi-projective moduli scheme of stable compactly supported sheaves, Hilbert schemes of proper sub-schemes or stable pairs.
- Have obstruction theory $\mathbb{E} = \tau_{[-2,0]}\big(\mathrm{RHom}(\mathcal{E},\mathcal{E})\big) \to \mathbb{L}_M$ resolved as

$$\mathbb{E} \cong (T \to E \to T^*) =: E^\bullet\,.$$

- Natural $E^\bullet \cong E_\bullet[2]$ induced by (E, q). The $O(n,\mathbb{C})$ structure of E reduces to $SO(n,\mathbb{C})$ iff the square root $\mathbb{Z}_2$-bundle O^M associated with $i^M : \det(\mathbb{E}) \xrightarrow{\sim} \det(\mathbb{E})^*$ is orientable.

Theorem

Let M be a quasi-projective moduli scheme of stable compactly supported sheaves, Hilbert schemes of proper sub-schemes or stable pairs on a quasi-projective Calabi–Yau 4-fold, then O^M is trivializable.

- Oh-Thomas use this to construct an isotropic cone $C_{E^\bullet} \subset E$ and define:

$$[M]^{\text{vir}} = \sqrt{0^!_E}[C_{E^\bullet}]\,, \qquad \hat{\mathcal{O}}^{\text{vir}}_M = \sqrt{0^*_E}[\mathcal{O}_{C_{E_\bullet}}]\sqrt{\det T^*}\,.$$

- **Example:** If M smooth, we have an obstruction bundle $\mathrm{Ob}(M)$ then $[M]^{\text{vir}} = e(\mathrm{Ob}_{\mathbb{R}}) = e(\Lambda)$ (if isotropic $\Lambda \subset \mathrm{Ob}$ exists).
- If X not compact still can construct $\hat{O}^{\text{vir}}_M$, if M compact have $[M]^{\text{vir}}$. Otherwise use localization formula to define invariants.
- Let T be a torus acting on X preserving ω_X ($\dim(T) \leq 3$), then action lifts to M of sheaves and $M^T \hookrightarrow M$. Have $E^\bullet = \{T \to E \to T^*\} \to \mathbb{L}_M$, then $E^\bullet|_{M^T} = E^\bullet_f \oplus (N^{\text{vir}})^\vee$.

Localization formula

- Under Serre-duality positive weights (with respect to some ordering) are paired with negative ones. Gives natural orientation on $(N^{\mathrm{vir}})^\vee$.
- **Using orientations in the non-compact setting**:
  1. If $E_f^\bullet \neq 0$, then orientation on $E^\bullet$ together with orientation of $(N^{\mathrm{vir}})^\vee$ induces one on $E_f^\bullet$ giving $[M^T]^{\mathrm{vir}}$.
  2. If $E_f^\bullet = 0$, then $(N^{\mathrm{vir}})^\vee = E^\bullet|_{M^T}$ and use the global orientation of $E^\bullet$ instead.
- Localization formulae:

$$[M]^{\mathrm{vir}} = i_* \frac{[M^T]^{\mathrm{vir}}}{\sqrt{e_T}(N^{\mathrm{vir}})}\,, \qquad \hat{\mathcal{O}}_M^{\mathrm{vir}} = i_* \frac{\hat{\mathcal{O}}_{M^T}^{\mathrm{vir}}}{\sqrt{e_T}(N^{\mathrm{vir}})}$$

- Used in the works of Cao, Kool, Maulik, Monavari, Nekrasov, Toda...

Global orientation for perfect complexes

- Toën–Vaquié(07') defined a functor $\mathcal{M}_{(-)} : \textbf{dg-Cat} \to \textbf{Hsta}_{\mathbb{C}}$. For X smooth quasi-projective $\mathcal{M}_X = \mathcal{M}_{L_{\mathrm{pe}}(X)}$, where $L_{pe}(X)$ the dg-category of perfect complexes. It classifies right proper object $\leftrightarrow$ compactly supported perfect complexes.
- We have $\mathbb{L}_{\mathcal{M}_X}$ the perfect cotangent complex. At a $\mathbb{C}$-point $[E^\bullet]$:

$$H^k(\mathbb{L}_{\mathcal{M}_X}|_{[E^\bullet]}) \cong \mathrm{Ext}^{1-k}(E^\bullet, E^\bullet)\,.$$

- Brav–Dyckerhoff(18') (PTVV for compact X) prove that $\boldsymbol{\mathcal{M}}_X$ is -2-shifted symplectic. Induces the Serre-duality isomorphism $\mathbb{L}_{\mathcal{M}_X} \xrightarrow{\sim} \mathbb{L}_{\mathcal{M}_X}[-2]$.
- Defining $K_{\mathcal{M}_X} = \det(\mathbb{L}_{\mathcal{M}_X})$ and using Serre duality, we get $i^\omega : K_{\mathcal{M}_X} \to (K_{\mathcal{M}_X})^*$. $O^\omega \to \mathcal{M}_X$ the square root $\mathbb{Z}_2$-bundle associated to i^ω.

Ambitious approach

Algebraic geometric	Differential geometric
compactly supported perfect complex $[E^\bullet]$	compactly supported pseudo-differential operator $\Psi : \Gamma^\infty_{\mathrm{cs}}(V_0) \to \Gamma^\infty(V_1)$
$\det(\mathrm{RHom}(E^\bullet, E^\bullet)) = \bigotimes_{i\in\mathbb{Z}} \det^{(-1)^i}\big(\mathrm{Ext}^i(E^\bullet, E^\bullet)\big)$	$\det(\Psi) = \det\big(\mathrm{Ker}(\Psi)\big)\det^*\big(\mathrm{Ker}(\Psi^*)\big)$
Serre-duality $\det(\mathrm{RHom}(E^\bullet, E^\bullet)) \cong \det(\mathrm{RHom}(E^\bullet, E^\bullet))^*$	Real structure $\Psi = \Psi_{\mathbb{R}} \otimes \mathrm{id}_{\mathbb{C}}$
$\mathrm{O}^\omega\|_{[E^\bullet]}$	$\mathrm{or}\big(\det_{\mathbb{R}}(\Psi_{\mathbb{R}})\big)$

Compactification

- Blanc(12') defines $(-)^{\mathrm{top}} : \mathbf{HSta}_{\mathbb{C}} \to \mathbf{Top}$
- For X a Hausdorff topological space, $\mathcal{C}_X = \mathrm{Map}\big((X^+, +), (BU \times \mathbb{Z}, 0)\big)$ classifies $K^0_{\mathrm{cs}}(X)$.
- Think of $\mathcal{C}_X$ as the differential geometric counter-part of $(\mathcal{M}_X)^{\mathrm{top}}$ with a natural map $\Gamma_X : (\mathcal{M}_X)^{\mathrm{top}} \to \mathcal{C}_X$.
- $K^0_{\mathrm{cs}}(X)$ can be expressed in terms of classes of (V_1, V_2, ϕ_∞), motivating what follows.
- Take $\bar{X}$ smooth projective compactification, s.t. $D = \bar{X} \backslash X$ is a strict normal crossing divisor.
- For any projective scheme Y define $\mathcal{M}^Y = \mathrm{Map}_{\mathbf{HSta}_{\mathbb{C}}}(Y, \mathrm{Perf})$.
- There is a natural map $\rho_D : \mathcal{M}_{\bar{X}} \to \mathcal{M}^D$ and we form $\mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}}$.

Notation:

- $\mathcal{E}\mathrm{xt}_L$ the complex on $\mathcal{M}_X$ given at a point $[E^\bullet]$ by $\mathrm{RHom}^\bullet(E^\bullet, E^\bullet \otimes L)$ for some coherent sheaf L (not necessarily compactly supported).
- We then have the duality $\mathcal{E}\mathrm{xt}_L \cong (\mathcal{E}\mathrm{xt}_{K_X \otimes L^\vee})^\vee[-4]$
- Set $\Lambda_L := \det(\mathcal{E}\mathrm{xt}_L)$, then

$$\Lambda_L \cong \Lambda^*_{K_X \otimes L^\vee} \, . \tag{1}$$

Definition

Let X be a smooth projective variety and K_X its canonical divisor. A divisor Θ, such that $2\Theta = K_X$ is called a *theta characteristic*. We say that (X, Θ) for a given choice of a theta characteristic Θ is *spin*. Applying (1) to the case $L = \Theta$, one obtains an isomorphism of line bundles $i^\Theta := i_\Theta : K_{\mathcal{M}_X} \to K^*_{\mathcal{M}_X}$. We define the orientation $\mathbb{Z}_2$-bundle O^Θ on $\mathcal{M}_X$ as the associated $\mathbb{Z}_2$-bundle to the isomorphism i^Θ.

Examples:

- $\mathbb{C}^4 \subset \mathbb{P}^1 \times \mathbb{P}^3$
- $\mathrm{Tot}(E \to V)$, s.t. $\det(E) = K_V$ and $\mathrm{rk}(E)$ odd, then take $\bar{X} = \mathbb{P}(E \oplus \mathcal{O}_S)$.
- $\mathrm{Tot}(L_1 \oplus Ł_2 \to S) \subset \mathbb{P}(L_1 \oplus \mathcal{O}_S) \times_S \mathbb{P}(L_2 \oplus \mathcal{O}_S)$
- Doesn't work if $\mathrm{rk}(E) = 2$ or for general toric CY 4-fold and will most likely depend on a choice of $\bar{X}$.

General approach:

- Inclusion $\zeta : \mathcal{M}_X \to \mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}}$, $[E^\bullet] \mapsto [i_* E^\bullet, 0]$.

Proposition

For a choice of extension data $\bowtie$, there exist a natural isomorphism $\vartheta_\bowtie : \pi_1^*(\Lambda_{\mathcal{O}_X}) \otimes \pi_2^*(\Lambda_{\mathcal{O}_X})^* \to \pi_1^*(\Lambda_{\mathcal{O}_X})^* \otimes \pi_2^*(\Lambda_{\mathcal{O}_X})$, such that its associated square root $\mathbb{Z}_2$-bundle $O^{\vartheta_\bowtie} \to \mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}}$ comes with a natural isomorphism $\zeta^*(O^{\vartheta_\bowtie}) \cong O^\omega$ on $\mathbb{Z}_2$-bundles on $\mathcal{M}_X$.

Construction of $O^{\vartheta_{\bowtie}}$, part 1

- Consider $[E^\bullet, F^\bullet, \phi] \in \mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}}$. Use notation $\det(E^\bullet, F^\bullet) = \det\big(\mathrm{RHom}(E^\bullet, F^\bullet)\big)$.
- Isomorphism:

$$\det(E^\bullet, E^\bullet)\det^*(F^\bullet, F^\bullet) \cong \det^*(E^\bullet, E^\bullet \otimes K_{\bar{X}})\det(F^\bullet, F^\bullet \otimes K_{\bar{X}})$$

- express $K_{\bar{X}} = \sum_{i=1}^N a_i D_i$, where D_i smooth irreducible divisors, s.t. $D = \bigcup_{i=1}^N D_i$.
- For each line bundle L have $0 \to L \xrightarrow{\cdot s_i} L(D_i) \to L(D_i)|_{D_i} \to 0$ which gives

$$\det\big(E^\bullet, E^\bullet \otimes L(D_i)\big)\det^*\big(F^\bullet, F^\bullet \otimes L(D_i)\big) \cong \det\big(E^\bullet, E^\bullet \otimes L\big)\det^*\big(F^\bullet, F^\bullet \otimes L\big)$$

Construction of $O^{\vartheta_{\bowtie}}$, part 2

- Repeat to obtain

$$\vartheta_{\bowtie} : \det(E^\bullet, E^\bullet)\det{}^*(F^\bullet, F^\bullet) \cong \det{}^*(E^\bullet, E^\bullet)\det(F^\bullet, F^\bullet)$$

- $O^{\vartheta_{\bowtie}}|_{[E^\bullet, F^\bullet, \phi]}$ square-root $\mathbb{Z}_2$-bundle associated to $\vartheta_{\bowtie}$.
- Extension data $\bowtie$ is collecting the data of the sections s_i and order of D_i used. One requires that $\prod(s_i)^{a_i}$ is a meromorphic extension of ω.

Now prove $O^{\vartheta_{\bowtie}}$ is trivializable and so $O^{\omega} \to \mathcal{M}_X$ is.

Main theorem

- Have the map $\Gamma : (\mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}})^{\mathrm{top}} \to \mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \mathcal{C}_{\bar{X}}$,

Theorem

Let X be a smooth Calabi–Yau 4-fold, $\bar{X}$ its smooth projective compactification by a strictly normal crossing divisor D. For any extension data $\bowtie$ the $\mathbb{Z}_2$-bundle

$$O^{\vartheta_{\bowtie}} \to \mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}} \tag{2}$$

is trivializable. There exists a natural trivializable $\mathbb{Z}_2$-bundle $D_O^{\mathcal{C}} \to \mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \mathcal{C}_{\bar{X}}$ with a natural isomorphism

$$\mathfrak{I}^{\bowtie} : \Gamma^*(D_O^{\mathcal{C}}) \cong (O^{\vartheta_{\bowtie}})^{top} \,. \tag{3}$$

Composing

$$\Gamma \circ \zeta^{\mathrm{top}} : (\mathcal{M}_X)^{\mathrm{top}} \to \mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \mathcal{C}_{\bar{X}} \,,$$

get a map that factors through $\Gamma_X : (\mathcal{M}_X)^{\mathrm{top}} \to \mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \{0\}$, where

$$\mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \{0\} = \mathrm{Map}_{C^0}\big((X^+, +), (BU \times \mathbb{Z}, 0)\big) = \mathcal{C}_X \,.$$

Theorem

Let (X, ω) be a quasi-projective Calabi–Yau 4-fold, then the $\mathbb{Z}_2$-bundle

$$O^\omega \to \mathcal{M}_X \tag{4}$$

is trivializable. Moreover, there is a canonical isomorphism

$$\mathfrak{I} : (\Gamma_X)^*(O^{cs}) \cong (O^\omega)^{top} \,.$$

- Can extend by the structure sheaf on D to get orientability of stable pair moduli spaces and Hilbert schemes.
- Let $\bar{\mathcal{M}}$ be a moduli stack of stable pairs or ideals sheaves on $\bar{X}$ with the projection $\pi_{\mathbb{G}_m} : \bar{\mathcal{M}} \to M$ which is a $[*/\mathbb{G}_m]$ principal bundle. We have an inclusion $\eta : \bar{\mathcal{M}} \to \mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}}$ given on points by mapping $[\bar{\mathcal{E}}] \mapsto ([\bar{\mathcal{E}}, \mathcal{O}_{\bar{X}}])$.

Theorem

Let $O^{\omega}_M \to M$ be the orientation bundle for M a moduli scheme of stable pairs or ideals sheaves of proper subschemes of X. There is a canonical isomorphism of $\mathbb{Z}_2$-bundles

$$\pi^*_{\mathbb{G}_m}(O^{\omega}_M) \cong \eta^*(O^{\vartheta_{\bowtie}})\,.$$

In particular, $O^{\omega}_M \to M$ is trivializable.

- Spaces $(\mathcal{M}_X)^{\mathrm{top}}$, $(\mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}})^{\mathrm{top}}$, $\mathcal{C}_X$, $\mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \mathcal{C}_{\bar{X}}$ are admissible H-spaces (in fact Γ-spaces/ E_∞-spaces), which are group-like

Definition (Cao–Gross–Joyce(18'))

X an H-space. A *weak H-principal* $\mathbb{Z}_2$*-bundle* on X is a $\mathbb{Z}_2$-bundle $P \to X$ with an isomorphism of $\mathbb{Z}_2$-bundles $p : P \boxtimes_{\mathbb{Z}_2} P \to \mu_X^*(P)$. A *strong H-principal* $\mathbb{Z}_2$*-bundle* on X is a pair (Q, q): trivializable $\mathbb{Z}_2$-bundle $Q \to X$, isomorphism of $\mathbb{Z}_2$ bundles

$$q : Q \boxtimes_{\mathbb{Z}_2} Q \to \mu_X^*(Q)\,,$$

such that under the homotopy $h : \mu_X \circ (\mathrm{id}_X \times \mu_X) \simeq \mu_X \circ (\mu_X \times \mathrm{id}_X) :$

$$(\mathrm{id}_X \times \mu_X)^*(q) \circ (\mathrm{id} \times q) : Q \boxtimes_{\mathbb{Z}_2} Q \boxtimes_{\mathbb{Z}_2} Q \to \big(\mu_X \circ (\mathrm{id}_X \times \mu_X)\big)^* Q$$

and

$$(\mu_X \times \mathrm{id}_X)^*(q) \circ (q \times \mathrm{id}) : Q \boxtimes_{\mathbb{Z}_2} Q \boxtimes_{\mathbb{Z}_2} Q \to \big(\mu_X \circ (\mu_X \times \mathrm{id}_X)\big)^* Q$$

$\mathbb{Z}_2$-graded H-principal $\mathbb{Z}_2$-bundles II

- A $\mathbb{Z}_2$-bundle $O \to X$ together with a continuous map $\deg(O) : X \to \mathbb{Z}_2$ is a $\mathbb{Z}_2$-*graded* $\mathbb{Z}_2$-*bundle*. If O_1, O_2 are $\mathbb{Z}_2$-graded then the isomorphism $O_1 \otimes_{\mathbb{Z}_2} O_2 \cong O_2 \otimes_{\mathbb{Z}_2} O_1$ differs by the sign $(-1)^{\det(O_1)\deg(O_2)}$ from the naive one.
- $\mathbb{Z}_2$-graded H-principal $\mathbb{Z}_2$-bundles combine the two definitions. Dual (O^*, p^*) defined by $O^* = O$ and $p^* = (-1)^{\deg(\pi_1^*(O))\deg(\pi_2^*(O))} p$. Isomorphisms have to preserve grading.

Examples:

- $\phi^\omega : O^\omega \boxtimes_{\mathbb{Z}_2} O^\omega \to \mu^*_{\mathcal{M}_X}(O^\omega)$, $\phi^{\vartheta_{\bowtie}} : O^{\vartheta_{\bowtie}} \boxtimes_{\mathbb{Z}_2} O^{\vartheta_{\bowtie}} \to \mu^*_{\mathcal{M}_{\bar{X},D}}(O^\omega)$ making them into weak H-principal $\mathbb{Z}_2$-bundles satisfying the associativity.
- Joyce–Tanaka–Upmeier(18') construct $\mathbb{Z}_2$-bundles $O_{\mathcal{C}}^{\not{D}_+} \to \mathcal{C}_X$ for X compact spin

Orientations via Dirac operators

- For any principal bundle P define the topological stack $\mathcal{B}_P = [\mathcal{A}_P/\mathcal{G}_P]$, $\mathcal{A}_P$ the space of connections, $\mathcal{G}_P$ the gauge group
- If Y is compact and spin, let $\not{D}_+ : S_+ \to S_-$ be the positive Dirac operator.
- Define $O_P^{\not{D}_+} \to \mathcal{B}_P$ by $O_P^{\not{D}_+}|_{[\nabla_P]} = \mathrm{or}\big(\det_{\mathbb{R}}(D^{\nabla}_{\mathrm{ad}(P)})\big)$ giving $O^{\not{D}_+} \to \mathcal{B}_Y = \bigcup_{[P]} \mathcal{B}_P$.
- There is a natural $\Sigma : (\mathcal{B}_Y)^{\mathrm{cla}} \to \mathcal{C}_Y$ which is a homotopy theoretic group completion of H-spaces. Using (weak) universality property get $O_{\mathcal{C}}^{\not{D}} \to \mathcal{C}_Y$.
- Cao–Gross–Joyce(18') prove that $O_{\mathcal{C}}^{\not{D}}$ is a strong H-principal $\mathbb{Z}_2$-bundle.
- The grading: $\deg(O_X^{\not{D}_+})|_{\mathcal{C}_\alpha} = \chi^{\not{D}_+}(\alpha, \alpha)$ where

$$\chi^{\not{D}_+}(E, E) = \mathrm{ind}(\not{D}^{\nabla_{\mathrm{ad}(P)}})$$

Orientations on the double

- Let $T \supset D$ be a tubular neighborhood (i.e. union of $T_i \supset D_i$), $K = X \backslash T$, $Y \subset X$ a manifold with a boundary containing K.
- $\tilde{Y} = Y \cup_Y (-Y)$ has a natural spin structure. Define $\tilde{T} = \bar{T} \cup (-Y)$.
- For each $P, Q \to \tilde{Y}$ pair of $U(n)$ bundles, s.t. $P|_{\tilde{T}} \cong Q|_{\tilde{T}}$. Consider $\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}}$ with an obvious action of $\mathcal{G}_P \times \mathcal{G}_Q$.
- Get the topological stack $\mathcal{B}_{P,Q,\tilde{T}} = [\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} / \mathcal{G}_P \times \mathcal{G}_Q]$

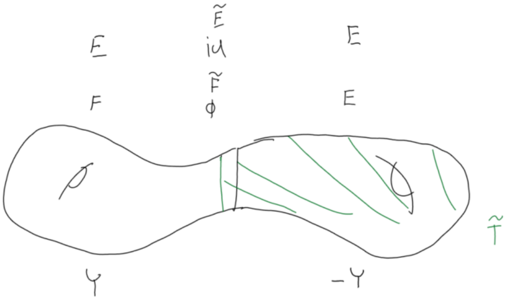
$$\mathcal{B}_{\tilde{Y},\tilde{T}} = \bigcup_{\substack{[P],[Q]: \\ [P|_{\tilde{T}}]=[Q|_{\tilde{T}}]}} \mathcal{B}_{P,Q,\tilde{T}} \,.$$

- Using $\mathcal{B}_{\tilde{Y}} \xleftarrow{p_1} \mathcal{B}_{\tilde{Y},\tilde{T}} \xrightarrow{p_2} \mathcal{B}_{\tilde{Y}}$ define

$$D_O(\tilde{Y}) = p_1^*(O^{\not{D}_+}) \boxtimes_{\mathbb{Z}_2} p_2^*((O^{\not{D}_+})^*) \,,$$

Transporting orientations back to $\bar{X}$

- Let $\mathcal{V}_Y = \mathrm{Map}_{C^0}(Y, Gr^\infty(\mathbb{C}))$, then $\mathcal{V}_{\tilde{Y}} \times_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_{\tilde{Y}} \simeq (\mathcal{B}_{\tilde{Y},\tilde{T}})^{\mathrm{cla}}$.
- Using a homotopy theoretic group completion $\mathcal{V}_{\tilde{Y}} \times_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_{\tilde{Y}} \to \mathcal{C}_{\tilde{Y}} \times_{\mathcal{C}_{\tilde{T}}} \mathcal{C}_{\tilde{Y}}$ get $D_O^{\mathcal{C}}(\tilde{Y})$ on the latter.
- Define $G_{\tilde{Y}} : \mathcal{V}_{\bar{X}} \times_{\mathcal{V}_D} \mathcal{V}_{\bar{X}} \to \mathcal{V}_{\tilde{Y}} \times_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_{\tilde{Y}}$, $[E, F, \phi] \mapsto [\tilde{E}, \tilde{F}, \tilde{\phi}]$



- Pullback $D_O(\tilde{Y})$ and $D_O^{\mathcal{C}}(\tilde{Y})$ to get D_O, $D_O^{\mathcal{C}}$.

Reducing to vector bundles

- For a scheme Z the moduli Ind-scheme of vector bundles generated by global sections $\mathcal{T}_Z = \mathrm{Map}_{\mathbf{IndSch}_{\mathbb{C}}}(Z, \mathrm{Gr}(\mathbb{C}^\infty))$,
- Have the homotopy commutative diagram of H-spaces

$$\begin{array}{ccc} \left(\mathcal{T}_{\bar{X}} \times_{\mathcal{T}_D} \mathcal{T}_{\bar{X}}\right)^{\mathrm{an}} & \xrightarrow{\Lambda} & \mathcal{V}_{\bar{X}} \times_{\mathcal{V}_D} \mathcal{V}_{\bar{X}} \\ \downarrow{\scriptstyle \Delta^{\mathrm{top}}} & & \downarrow{\scriptstyle \Omega} \\ (\mathcal{M}_{\bar{X}} \times_{\mathcal{M}^D} \mathcal{M}_{\bar{X}})^{\mathrm{top}} & \xrightarrow{\Gamma} & \mathcal{C}_{\bar{X}} \times_{\mathcal{C}_D} \mathcal{C}_{\bar{X}} \end{array} \quad . \qquad (5)$$

- Δ^{top} and Ω are homotopy theoretic group completions $\implies$ only need to construct a natural isomorphism $\Lambda^*(D_O) \cong (\Delta^{\mathrm{top}})^*(O^{\vartheta_{\bowtie}})$ and show it is a strong H-principal $\mathbb{Z}_2$-bundle isomorphism to get $\Gamma^*(D_O^{\mathcal{C}}) \cong O^{\vartheta_{\bowtie}}$

Differential geometric side:

- Given by $D_O|_{[E,F,\phi]} = \mathrm{or}\big(\det_{\mathbb{R}}(\not{D}_+^{\nabla_{\mathrm{ad}(\tilde P)}})\big) \otimes_{\mathbb{Z}_2} \mathrm{or}\big(\det^*_{\mathbb{R}}(\not{D}_+^{\nabla_{\mathrm{ad}(\tilde Q)}})\big)$, where $\tilde P, \tilde Q$ associated $U(n)$ bundles to $\tilde E$, $\tilde F$.
- Symbol map (Atiyah–Singer(71')) $\sigma : \Psi DO_m(E_0, E_1) \to \mathrm{Sym}_m(E_0, E_1)$, then $\sigma(\not{D}_+^{\nabla_{\mathrm{ad}(P)}}) = \sigma(\not{D}_+) \otimes \mathrm{id}_{\pi^*(\mathrm{ad}(P))}$, where $\pi : T\tilde Y \to \tilde Y$.
- Elliptic symbols of degree m: $\mathrm{Ell}_m(E_0, E_1)$. There is a map $(-)_0 : \mathrm{Ell}_m(E_0, E_1) \to \mathrm{Ell}_0(E_0, E_1)$.
- $\mathrm{or}(-)$ depends only on $\sigma(D)$ and $\mathrm{or}\big(\sigma(D)\big) = \mathrm{or}\big((\sigma(D))_0\big)$
- Using deformation of symbols in families (Upmeier(19'), Donaldson–Kronheimer) get

$$\mathrm{or}(\Psi_{\mathbb{R}}) = \mathrm{or}\begin{pmatrix} \chi\sigma(\not{D}_+)_0 \otimes \mathrm{id}_{\pi^*(\mathrm{ad}(P)} & (1-\chi)\mathrm{ad}(\phi)^{-1} \\ (1-\chi)\mathrm{ad}(\phi) & -\chi\big(\sigma(\not{D}_+)_0 \otimes \mathrm{id}_{\pi^*(\mathrm{ad}(Q))}\big)^* \end{pmatrix}$$

Idea of the proof

Algebraic geometric side:

- For simplicity assume $K_{\bar{X}} = D_1$
- Recall that we used $0 \to \mathrm{End}(E) \xrightarrow{\cdot s_i} \mathrm{End}(E)(D_1) \to \mathrm{End}(E)(D_1)|_{D_1} \to 0$ (+same for $\mathrm{End}(F)$).
- Replace $\mathrm{End}(E)(D_1)|_{D_1}$ by a common resolutions: $\mathrm{End}(E) \oplus \mathrm{End}(F) \to K$, where $K = \ker\Big(\mathrm{End}(E)(D_1) \oplus \mathrm{End}(F)(D_1) \to \mathrm{End}(E)(D_1)|_{D_1}\Big)$
- Express everything using vector bundles and their Dolbeault resolutions.

Comparing both sides

- Use deformation of complex determinant line bundles of symbols up to (contractible) isotopy to deform symbols of Dolbeault operator into compactly supported Ψ and express their the algebraic isomorphism as a real structure $\Psi_{\mathbb{R}}$ (see https://arxiv.org/abs/2008.08441)

Compatibility under direct sums

- $\mathcal{C}_\alpha$ connected component of $\mathcal{C}_X$ corresponding to $\alpha \in K^0_{\mathrm{cs}}(X)$ and $O^{\mathrm{cs}}_\alpha = O^{\mathrm{cs}}|_{\mathcal{C}_\alpha}$
- $\mu_{\mathcal{C}} : \mathcal{C}_X \times \mathcal{C}_X \to \mathcal{C}_X$
- There are natural isomorphisms $\tau^{\mathrm{cs}} : O^{\mathrm{cs}} \boxtimes_{\mathbb{Z}_2} O^{\mathrm{cs}} \to \mu^*_{\mathcal{C}}(O^{\mathrm{cs}})$ and $\phi^\omega : O^\omega \boxtimes_{\mathbb{Z}_2} O^\omega \to \mu^*_{\mathcal{M}_X}(O^\omega)$.
- Could choose trivializations o^{cs}_α of O^{cs}_α. These induce $o^\omega_\alpha = \mathfrak{I}\big((\Gamma^{\mathrm{cs}})^*(o^{\mathrm{cs}}_\alpha)\big)$ orientations of O^ω_α which is the restriction of O^ω to $\mathcal{M}_\alpha = \Gamma^{-1}(\mathcal{C}_\alpha)$.
- We can ask about how these orientations behave under addition : Important for constructing natural orientations and Joyce's vertex algebra used to express WCF.

Theorem

For all $\alpha, \beta \in K^0_{cs}(X)$*:* $\tau^{cs}_{\beta,\alpha} = (-1)^{\bar{\chi}(\alpha,\alpha)\bar{\chi}(\beta,\beta)+\bar{\chi}(\alpha,\beta)}\tau^{cs}_{\alpha,\beta}$*, where* $\bar{\chi} : K^0_{cs}(X) \times K^0_{cs}(X) \to \mathbb{Z}$ *is the compactly supported Euler form. For all* $\alpha\beta \in K^0_{cs}(X)$*, then there are* $\epsilon_{\alpha,\beta} \in \{-1, 1\}$*, defined by* $\tau^{cs}_{\alpha,\beta}(o^{cs}_\alpha \boxtimes_{\mathbb{Z}_2} o^{cs}_\beta) = \epsilon_{\alpha,\beta}\, \mu^*_{cs}(o^{cs}_{\alpha+\beta})$*, such that they satisfy* $\epsilon_{\beta,\alpha} = (-1)^{\bar{\chi}(\alpha,\alpha)\bar{\chi}(\beta,\beta)+\bar{\chi}(\alpha,\beta)}\epsilon_{\alpha,\beta}$ *.*
Same can be said for o^ω_α*.*

Summary

- All reasonable moduli spaces (compactly supported perfect complexes, Hilbert schemes, stable pairs) are orientable.
- These orientations are pullbacks of differential geometric ones which are compactly supported in X.
- They satisfy relations under sums which make them compatible with the vertex algebras on $H_*(\mathcal{M}_X)$.