Towards global homological mirror symmetry for genus 2 curves

joint w/ H. Azam, H. Lee, and C-C.M. Liu

Main result

my PhD thesis ⇒ ∃ functor $D^b Coh(\Sigma_2) \xrightarrow{\cong} \mathcal{L} \subseteq H^0 FS(\underbrace{Y, v_0})$

1 param family of genus 2 curves

1 param family of s. fibrations $v_0 : Y \to \mathbb{C}$

w/ Azam, Lee, & Liu ⇒ can upgrade and adapt result to 6 parameters describing the moduli space of cx strs on Σ_2 and of s. strs on (Y, v_0)

Talk outline

1. HMS background (T^2)
2. PhD thesis result
3. ACLL

---

§1 HMS background

Zaslow-Polishchuk: "Categorical mirror symmetry for the elliptic curve"

A-side = symplectic	B-side = complex
$(T^2, \int_{T^2} \omega = a \in (0,\infty))$	$(\mathbb{C}/\mathbb{Z}^2,\ z \sim z+1,\ z \sim z+\tau)$
	$\tau = i \cdot a \in$ upper half plane

Remark

$\dim_{\mathbb{R}}$ (moduli space of cx strs on $\mathbb{C}/\mathbb{Z}^2$) = 2

Remark $\dim_{\mathbb{R}}$ (moduli space of cx strs on $\mathbb{C}/\mathbb{Z}^2$) $= 2$

analogue on A-side is "B-field" $\omega \rightsquigarrow \omega + ib$
allows us to vary $\tau \in$ ///////

Geometrically: $\left(T^2, \int_{T^2} \omega = a\right) \overset{}{\longleftrightarrow}_{\text{mirror}} \left(\mathbb{C}/\mathbb{Z}^2, \begin{smallmatrix} z \sim ia \\ z \sim 1 \end{smallmatrix}\right)$

Algebraically $D^b \operatorname{Coh}(\mathbb{C}/\mathbb{Z}^2) \cong \underbrace{\operatorname{Fuk}(T^2)}_{(*)}$

(*) objects = closed Lagrangians (up to Hamiltonian isotopy)

Lagr $= \frac{1}{2}$ dim^l submanifold on which ω vanishes
$=$ closed curves

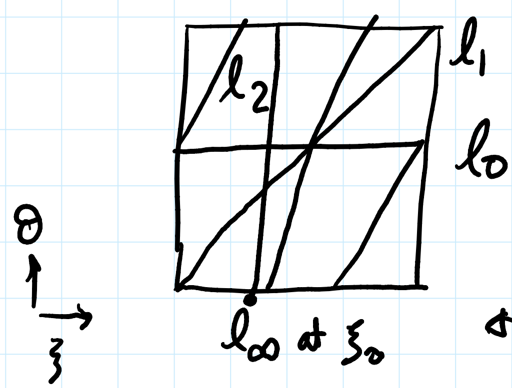
non-deg closed 2-form

Hamiltonian: $\iota_{X_H} \omega = d\underbrace{H}_{\text{smooth}}$

flow ϕ^t

L and $\phi^t(L)$ are equivalent objects in Fuk cat

here: objects = rational slope lines



$\longleftrightarrow$ object in $D^b \operatorname{Coh}$, namely $\mathcal{O}_{z_0}$ s.t. $\log |z_0| = \xi$

(varying z_0 corresponds to adding a local system as part of the data on Lagr: flat unitary connxn on $l_0 \times \mathbb{C}$, ie $e^{i\theta} \in S^1$)

morphisms of $\operatorname{Fuk}(T^2)$ should match w/ $\operatorname{Ext}_{\mathbb{C}/\mathbb{Z}^2}(\mathcal{L}^i, \mathcal{L}^j)$

where $\mathcal{L} \to$ is a line bundle of degree 1.

where $\mathcal{L} \to$ is a line bundle of degree 1.

Claim is that $\ell_1 \xleftrightarrow[\text{objects}]{\text{HMS}} \mathcal{L}$, need $\mathrm{Ext}_{\mathbb{C}/\mathbb{Z}^2}(\mathcal{L}^i, \mathcal{L}^j) \cong \mathrm{Hom}_{\mathrm{Fuk}}(\ell_i, \ell_j)$

$$\ell_i \longleftrightarrow \mathcal{L}^{\otimes i}$$

$$\underbrace{\mathrm{Hom}_{\mathrm{Fuk}}(\ell_i, \ell_j) = HF(\ell_i, \ell_j)}_{\text{homology}} = \underbrace{\bigoplus_{p \in \ell_i \cap \ell_j} \mathbb{C} \cdot p}_{\text{Chain cx}}$$

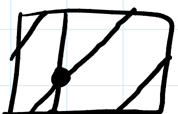
the $\partial = 0$ (it counts bigons, analogous how we will count triangles in a moment)

Input $\cap$ pt $\rightsquigarrow$ $\boxed{\partial}$ $\rightsquigarrow$ sum over $\cap$ pts

Input 2 $\cap$ pts $\rightsquigarrow$ $\boxed{\text{composition of mrs}}$ $\rightsquigarrow$ sum over $\cap$ pts $\Big\}$ triangles between 3 $\cap$ pts

Why should $HF(\ell_i, \ell_j) \cong \mathrm{Ext}(\mathcal{L}^i, \mathcal{L}^j)$?

Family Fiber theory



$\ell_{i \in \mathbb{Z}}$

$(\xi_0, i \cdot \xi_0) \longmapsto \ell_\infty^{\xi_0}$

is $\cap$ pt

$\ell_i \cap \ell_\infty = 1$ pt

$\mathrm{Ext}(\mathcal{L}, \mathcal{O}_{z_0}) \cong \mathcal{L}|_{z_0}$

$\quad \|$

$HF(\ell_i, \ell_\infty) = \mathbb{C} \cdot p$

Idea defining $\mathcal{L}|_{z_0} := \mathbb{C} \cdot p$, one can put a holo str of a degree 1 line bundle on $\mathcal{L}$ (Fukaya: HMS for abelian varieties).

Upshot
$$D^b\mathrm{Coh}(\mathbb{C}/\mathbb{Z}^2) \to \mathrm{Fuk}(T^2)$$
$$\mathcal{L}^{\otimes i} \to \ell_i$$
$$H^0(\mathcal{L}^{j-i}) \cong \mathrm{Ext}(\mathcal{L}^i, \mathcal{L}^j) \cong HF(\ell_i, \ell_j)$$

$j > i$
$$(s_1, \dots, s_{j-i}) \longmapsto (p_1, \dots, p_{j-i})$$

$j-i$ rank, isomor as vec spaces

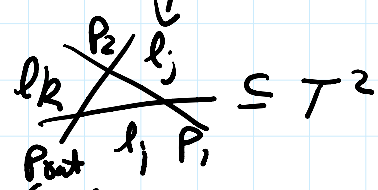
Is this a functor ie is composition respected?

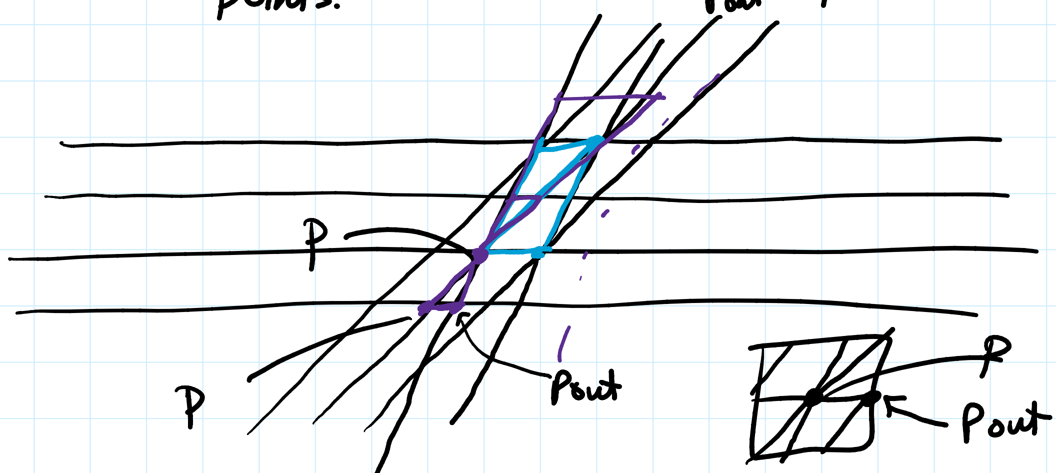
Is this a functor i.e. is composition respected?

Fuk) $HF(\ell_j, \ell_k) \otimes HF(\ell_i, \ell_j) \to HF(\ell_i, \ell_k)$ $\quad \ell_i \to \ell_j \to \ell_k$

$$(p_2, p_1) \longmapsto \sum_{p_{out} \in \ell_i \cap \ell_k} \# \left(\begin{matrix} \text{i-holo}^c \text{ triangles} \\ D \ni 3 \text{ pts} \end{matrix} \right) e^{-\int_\Delta \omega} \cdot p_{out}$$

Riem map thm ⇒ only 1 triangle b/w any 3 candidate points.





adding up areas we find an infinite sum ⇝ theta functions

Functor is indeed a functor:

$i=0$
$j=1$
$k=2$

$$\underset{Ext(\mathcal{L}, \mathcal{L}^2)}{s} \cdot \underset{Ext(\mathcal{O}, \mathcal{L})}{s} = \boxed{c_1}\, s_1 + \boxed{c_2} \cdot s_2$$

$s_1, s_2 \in H^0(\mathcal{L}^2)$

$$p \cdot P = \boxed{c_1} \cdot p + \boxed{c_2} \cdot p_{out}$$

§2: Thesis result

Step 1 Upgrade above T^4. (prev. known by Fukaya's HMS on abelian varieties)

Step 1 Upgrade above T. (prev. known by Fukaya's HMS on abelian varieties)

A-side	B-side
$V^\vee := \mathbb{R}^4/\mathbb{Z}^4$	$V := (\mathbb{C}^*)^2/\Gamma_B$

$\Gamma_B = \mathbb{Z}\langle \binom{2}{1}, \binom{1}{2} \rangle$

$= \{(\xi_1, \xi_2, \theta_1, \theta_2)\}$

$\xi \in \mathbb{R}^2/\Gamma_B = T_B$

$\theta \in \mathbb{R}^2/\mathbb{Z}^2 = T_F$

$V \supset \Sigma_2$ genus 2 curve,

$\Sigma_2 = s^{-1}(0)$

$$\begin{tikzcd} \mathcal{L} \arrow[d] \\ V \arrow[u, bend right, "s"'] \end{tikzcd}$$

$l_k := \left\{ \left(\xi_1, \xi_2, -k \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right) \right\}$ $(\xi_1, \xi_2) \in T_B$

12 T^2

$\gamma = \gamma_1 \cdot \binom{1}{0} + \gamma_2 \cdot \binom{0}{1}$

// How Γ_B acts: on $(\mathbb{C}^*)^2$ $\gamma \cdot (x_1, x_2) = (t^{-\gamma_1} x_1, t^{-\gamma_2} x_2)$

t 1-param family

$\mathcal{L} = (\mathbb{C}^*)^2 \times \mathbb{C} / \Gamma_B$ $\quad (x_1, x_2, v) \sim (\gamma \cdot (x_1, x_2), x^{\lambda(\gamma)} t^{\chi(\gamma)} \cdot v)$

$\downarrow$

V

determines 1st Chern class: $\lambda : \binom{2}{1} \mapsto \binom{1}{0}$, $\binom{1}{2} \mapsto \binom{0}{1}$ extend $\mathbb{Z}$-linearly

determines the holo^c str: $\chi(\gamma) := -\frac{1}{2} \langle \gamma, \lambda(\gamma) \rangle$

$l_j \mapsto \mathcal{L}^{\otimes i}$ homs match as vec. spaces: dim $(j-i)^2$

respects composition: a triangle count and multiplication by theta fns = sections of $\mathcal{L}^{\otimes j-i}$

Step 2 $T^4 \rightsquigarrow \Sigma_2$: "smaller on B-side, larger on A-side"

Step 2 $T^4 \rightsquigarrow \Sigma_2$: "smaller on B-side, larger on A-side"


on a torus

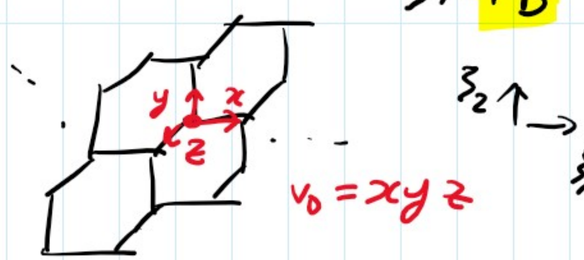
$$\begin{array}{ccccc} \mathcal{L}^{\otimes i} & D^b Coh(V) & \xrightarrow{i^*} & D^b Coh(\Sigma_2) & \mathcal{L}^i|_{\Sigma_2} \\ \downarrow & \downarrow{\scriptstyle (\text{Fukaya})} & \circlearrowright & \downarrow{\scriptstyle (\text{C.})} & \downarrow{\scriptstyle \text{fully-faithful embedding}} \\ l_i & H^0 Fuk(V^\vee) & \xrightarrow{U} & \underbrace{H^0 FS(Y, v_0)} & L_i \end{array}$$

What is (Y, v_0)? Abouzaid-Auroux-Katzarkov found mirrors to hypersurfaces of toric vars: in this case

$$\underbrace{\Delta_Y}_{\text{(generalized SYZ mirror to } \Sigma_2)} = \left(\bigcap_{\gamma \in \Gamma_B} \left\{ y := \underbrace{(y_1, y_2, y_3)}_{\text{notation used more universally as coords on polytope}} = (\xi_1, \xi_2, \eta) \in \mathbb{R}^3 \;\middle|\; l_\gamma(y) := \langle y, \underbrace{\begin{pmatrix} -\lambda(\gamma) \\ 1 \end{pmatrix}}_{\text{inward normal}} \rangle - \kappa(\gamma) \geq 0 \right\} \right) \Big/ \Gamma_B$$

toric variety of infinite type / Γ_B-action

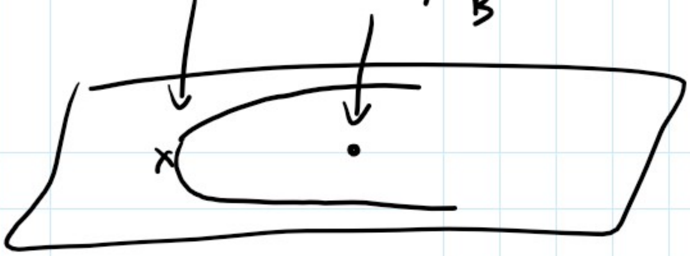
what (Y, v_0) looks like:



$v_0 = xyz$

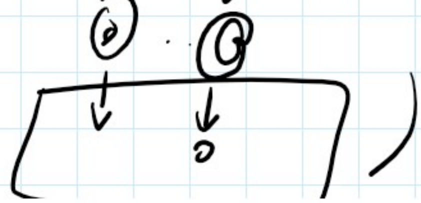
toric degen

$l_k \subseteq T^4$ $\square / \Gamma_B$



$\bigcup_{\text{U-shape}} l_k =: L_k \in FS(Y, v_0)$

(analogue of this for pt $\subseteq$ ellip curve & gener. SYZ


)

shape

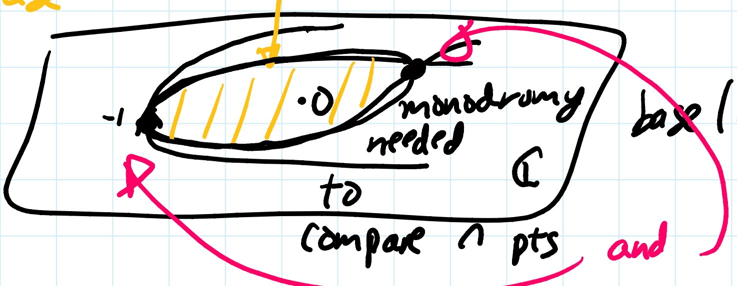
Main computation: morphism groups match $\mathrm{Ext}(\mathcal{L}^i|_{\Sigma_2}, \mathcal{L}^j|_{\Sigma_2})$

$$\begin{tikzcd} \mathcal{L} \arrow[d] \\ V \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \Sigma_2 = s^{-1}(0)$$

$$\begin{tikzcd} H^0(\mathcal{L}^{j-i-1}) \otimes S \arrow[r] \arrow[d, "\cong"', "\text{HMS for } T^4"] & H^0(\mathcal{L}^{j-i}) \arrow[r] \arrow[d, "\cong"] & H^0(s_* \mathcal{O}_{\Sigma_2} \otimes \mathcal{L}^{j-i}) \arrow[r] \arrow[d, "\cong"] & 0 \\ CF(l_{i+1}, l_j) \arrow[r, "\partial"] & \underbrace{CF(l_i, l_j)}_{\text{in a } T^4 \text{ fiber}} \arrow[r] & HF(L_i, L_j) \arrow[r] & 0 \end{tikzcd}$$

∂ = counts of holo bigons over base

$\partial = (\underbrace{OGW_1}_{KL}) \cdot s$



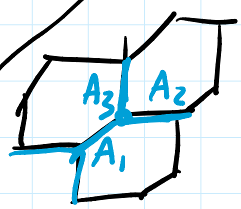
base (V_0)

§3: ACLL work

$i \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightsquigarrow \tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \in$ Siegel space

cx str on V

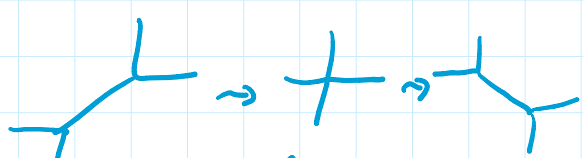
$$\underbrace{\begin{pmatrix} A_1 + A_2 & A_1 \\ A_1 & A_1 + A_3 \end{pmatrix}}_{(A_1, A_2, A_3, B_1, B_2, B_3)}$$



areas of $\mathbb{P}^1$'s

wall-crossing behavior

shrink A_i's to 0



$A_1 = 0$

- incorporate B-field
- use ω from sympl reduction

parallel transport? does ...

$A_1 = 0$

parallel transport, monodromy } done in (ζ, θ) coords

- wall-crossing behavior $\underbrace{q_j}_{\text{Kähl parameters}} = t^{2\pi i (A_j + i B_j)}$

Using modular property $\underset{\theta\text{-fn}}{s}(Az \mid A\tau \underset{GL_2\mathbb{Z}}{A^t}) = s(z \mid \tau)$

Lemma Recall above $\partial \mathcal{L} \otimes s$. ∂ is a (disc count). $\underbrace{\text{(sphere}}_{KL}$ count) and disc count $= S$. s is invt under the following transformation:

$$\hat{z}_1 = z_1^a z_2^b, \quad \hat{z}_2 = z_1^c z_2^d, \quad \hat{z}_3 = z_3$$

$$\hat{q}_1 = q_1^{ac+bd+ad+bc} q_2^{ac} q_3^{bd}$$

$$\hat{q}_2 = q_1^{(a+b)^2 - ac - bd - ad - bc} q_2^{a^2 - ac} q_3^{b^2 - bd}$$

$$\hat{q}_3 = q_1^{(c+d)^2 - ac - bd - ad - bc} q_2^{c^2 - ac} q_3^{d^2 - bd}$$

ie $S(\hat{z}, \hat{q}) = S(z, q)$ $\leftrightarrow$ global HMS

$$= z_3 \sum_{n \in \mathbb{Z}^2} z^n (q_1 q_2)^{\frac{n_1^2}{2}} (q_1 q_3)^{\frac{n_2^2}{2}} q_1^{n_1 n_2}$$

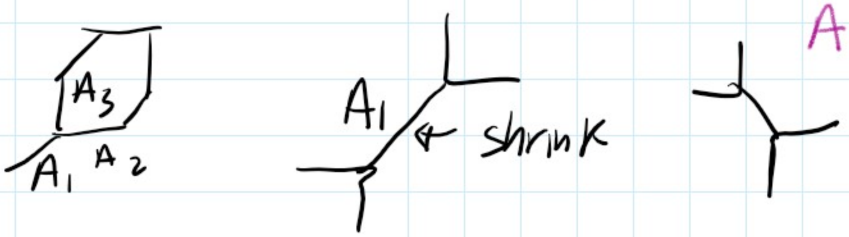
Change of formula arises from modular transformation property $S(Az \mid A\tau A^t) = S(z \mid \tau)$.

$A \in GL_2(\mathbb{Z})$

Corollary

Corollary

$S(Az \mid A\tau A^t) = S(z \mid \tau)$.
$A \in GL_2(\mathbb{Z})$



passing through each $A_i = 0$ is a transformation of the above form.

So the Θ-fn is globally defined.

2. (Switching A_2 and A_3) $(a,b,c,d) = (1,-1,0,-1)$

$$\hat{z}_1 = z_1 z_2^{-1}, \quad \hat{z}_2 = z_2^{-1}, \hat{z}_3 = z_3, \quad \hat{q}_1 = q_3, \quad \hat{q}_2 = q_2, \quad \hat{q}_3 = q_1.$$

3. (Switching A_1 and A_3) $(a,b,c,d) = (0,1,1,0)$

$$\hat{z}_1 = z_2, \quad \hat{z}_2 = z_1, \quad \hat{z}_3 = z_3, \quad \hat{q}_1 = q_1, \quad \hat{q}_2 = q_3, \quad \hat{q}_3 = q_2.$$

4. (Switching A_1 and A_2) $(a,b,c,d) = (-1,0,-1,1)$

$$\hat{z}_1 = z_1^{-1}, \quad \hat{z}_2 = z_1^{-1} z_2, \quad \hat{q}_1 = q_2, \quad \hat{q}_2 = q_1, \quad \hat{q}_3 = q_3.$$