Combinatorial Mutations and Block Diagonal Polytopes

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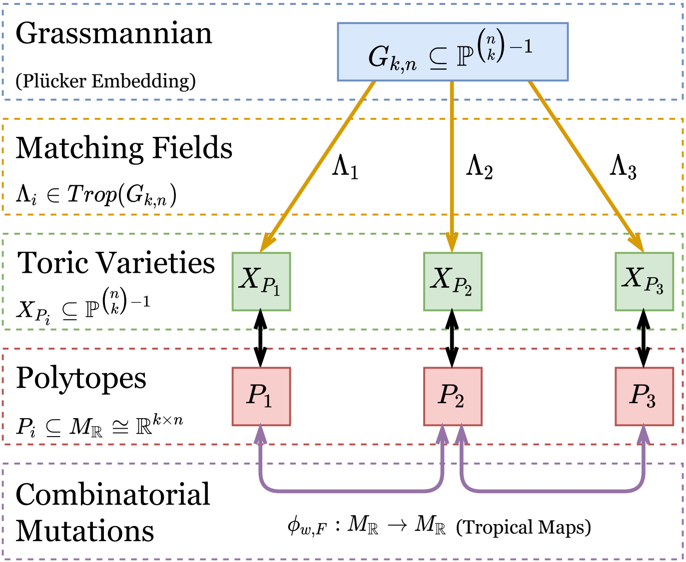
12th August 2021

The Big Picture



Outline

1. *Polytopes and Combinatorial Mutation*
2. *Toric Varieties and Toric Degeneration*
3. *Our Results*

1. *Polytopes and Combinatorial Mutation*

Background on Combinatorial Mutations

Minkowski Polynomials and Mutations (ACGK-12).

- A mirror partner $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, for a Fano manifold X, encodes Gromov-Witten invariants of X.
- **Combinatorial mutations** of Laurent polynomials connect polynomials with the same classical period. The induced map on their Newton polytopes gives rise to mutations of polytopes.

Mutations of Laurent Polynomials and Flat Families with Toric Fibers (Ilten-12).

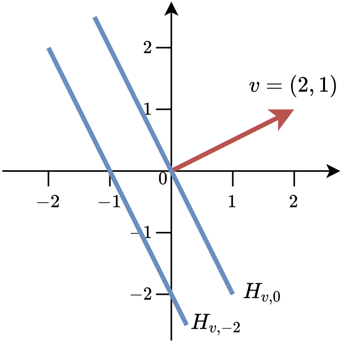
- If P and Q are polytopes related by a combinatorial mutation then there exists a flat family $X \to \mathbb{P}^1$ such that $X_0 \cong X_P$ and $X_\infty \cong X_Q$.

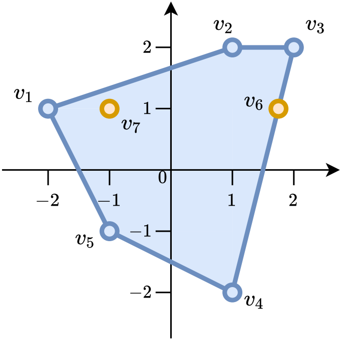
Wall-Crossing for Newton-Okounkov bodies (EH-20).

- Combinatorial mutations connect Newton-Okounkov bodies arising from adjacent cones in the tropicalization of $G_{2,n}$.

Notation

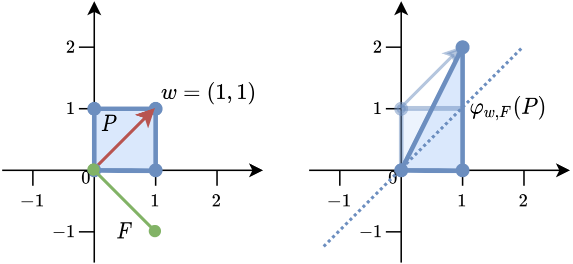
- **Euclidean vector space** $E = (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \supseteq \mathbb{Z}^n$ and **lattice**.
- **Lattice polytope** $\mathrm{Conv}(v_1, \dots, v_k) \subseteq E$ where $v_i \in \mathbb{Z}^n$.
- **Hyperplane** $H_{v,h} = \{x \in E : \langle x, v \rangle = h\}$.
- **Primitive lattice point** $(a_1, \dots, a_n) \in \mathbb{Z}^n$ if $\gcd(a_1, \dots, a_n) = 1$.





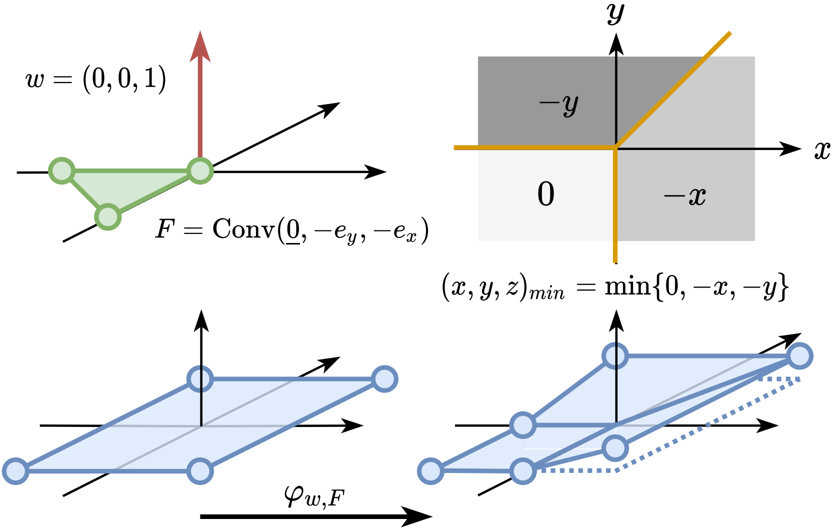
Tropical Maps and Combinatorial Mutation

- $w \in \mathbb{Z}^n \subseteq E$ **primitive** lattice point.
- $F \subseteq H_{w,0}$ **lattice polytope** in the orthogonal space to w.
- **Tropical map** $\varphi_{w,F} : E \to E : x \mapsto x - x_{\min} w$ where $x_{\min} = \min\{\langle x, f \rangle : f \in F\}$.
- **Combinatorial mutation** $P \mapsto \varphi_{w,F}(P)$ if the image is convex.



A tropical map is a piecewise shear.

Tropical Map Example



Ehrhart Polynomial

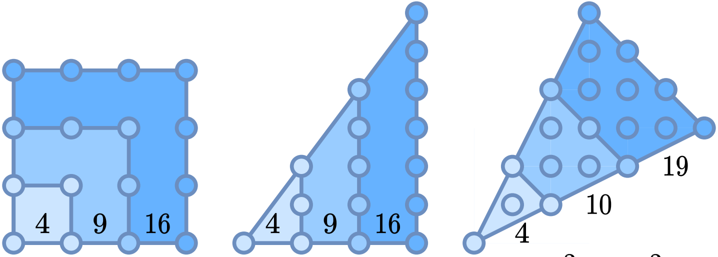
- $P = \text{Conv}(v_1, \dots, v_k)$ a lattice polytope of dimension d.
- **nth dilation of** P: $nP = \text{Conv}(nv_1, \dots, nv_k)$.

Theorem / Definition (Ehrhart Polynomial).

There exists a degree d polynomial $L_P \in \mathbb{Q}[x]$ such that for all $n \in \mathbb{N}$

$$L_P(n) = |\{w \in \mathbb{Z}^n : w \in nP\}|.$$

- Combinatorial mutation preserves the Ehrhart polynomial [ACGK12].



$L_P(x) = (x+1)^2$

$L_P(x) = \frac{3}{2}x^2 + \frac{3}{2}x + 1$

The Polytopes of a Poset

Let Π be a finite set with partial order $\prec$.

- **Order polytope**:

$$\mathcal{O}(\Pi) = \mathrm{Conv}\{(x_p) \in \mathbb{R}^\Pi : 0 \leq x_p \leq x_q \leq 1 \text{ if } p \preceq q \text{ in } \Pi\}.$$

- **Chain polytope**:

$$\mathcal{C}(\Pi) = \mathrm{Conv}\{(x_p) \in \mathbb{R}^\Pi : x_{p_{i_1}} + \cdots + x_{p_{i_k}} \leq 1 \text{ if } p_{i_1} \prec \cdots \prec p_{i_k} \text{ in } \Pi \text{ and } x_p \geq 0 \text{ for all } p \in \Pi\}.$$

Theorem (Higashitani 2020).

There exists a sequence of combinatorial mutations taking $\mathcal{O}(\Pi)$ to $\mathcal{C}(\Pi)$. In particular, they have the same Ehrhart polynomial.

- We say such polytopes are **mutation equivalent**.
- Example: *GT*-polytope and *FFLV*-polytope are mutation equivalent.

Questions

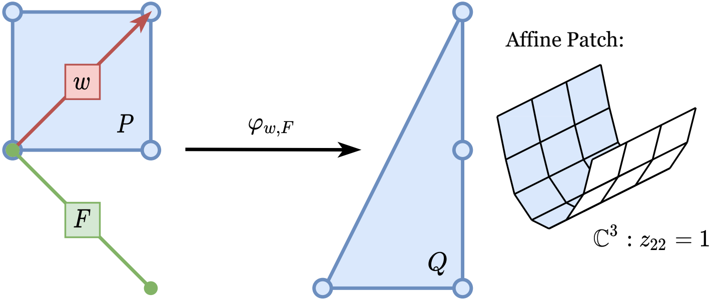
- Which properties are shared by *mutation equivalent polytopes*?
- What is the relationship between their *toric varieties*?

$$X_P \cong \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$$

$$\langle z_{12} z_{34} - z_{13} z_{24} \rangle \subseteq \mathbb{C}[z_{12}, z_{13}, z_{24}, z_{34}]$$

$$X_Q \subseteq \mathbb{P}^3 \qquad \textit{"Parabolic Cylinder"}$$

$$\langle z_{22} z_{33} - z_{23}^2 \rangle \subseteq \mathbb{C}[z_1, z_{22}, z_{23}, z_{33}]$$



2. *Toric Varieties and Toric Degeneration*

Toric Degenerations

A *toric degeneration* (of a variety X) is a flat family $\mathcal{F}_t \to \mathbb{A}^1$ such that $\mathcal{F}_0$ is a toric variety and all other fibers $\mathcal{F}_t$ where $t \neq 0$ are isomorphic (to X).

- If X is a variety and $\mathcal{F}$ is toric degeneration, then some algebraic invariants of X can be read from any fiber, in particular the toric fiber.
- Toric varieties are well studied and many of their algebraic invariants can be given combinatorially in terms of their polytope.

Questions.

- What are the toric degenerations of a given variety X?
- What structures exist to parametrise toric degenerations?

Grassmannians

- **Grassmannian** $G_{k,n}$, the set of k-dimensional linear subspaces of $\mathbb{C}^n$.
- Under the **Plücker embedding**, $G_{k,n} \subseteq \mathbb{P}^{\binom{n}{k}-1}$ is the vanishing set of the Plücker ideal $I_{k,n} \subseteq \mathbb{C}[P_J : J \in \binom{[n]}{k}]$. The ideal is

$$I_{k,n} = \ker(\mathbb{C}[P_J] \to \mathbb{C}[x_{i,j}] : P_J \mapsto \det(X_J))$$

where $\det(X_J)$ is a maximal minor of a $k \times n$ matrix of variables.

Example: $G_{2,4}$.

Let $X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$.

The ideal $I_{2,4}$ is the kernel of the map $P_{ij} \mapsto (x_i y_j - x_j y_i)$.

The Plücker ideal is:

$$I_{2,4} = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle.$$

A toric degeneration of $G_{2,4}$ is $\mathcal{F}_t$ where

$$\mathcal{F}_t = V(tP_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23}),$$

$$\mathcal{F}_0 = V(P_{13}P_{24} - P_{14}P_{23}).$$

Gröbner Degeneration

- $R = \mathbb{C}[x_1, \dots, x_n]$ polynomial ring and $w \in \mathbb{R}^n$ a weight vector for R.
- $f = \sum_{u \in \mathbb{N}^n} c_u x^u \in R$ polynomial.
- $\mathrm{in}_w(f) = \sum_u c_u x^u$ **lead term of** f, where the sum is taken over $u \in \mathbb{N}^n$ such that $c_u \neq 0$ and $u.w$ is minimum.

- $I \subseteq R$ ideal.
- $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$ **initial ideal**. There exists a flat family whose special fiber is $V(\mathrm{in}_w(I))$.

Example.

- $R = \mathbb{C}[P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}]$
- $w = (1, 0, 0, 0, 0, 0) \in \mathbb{R}^6$
- $I = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle \subset R$

The initial ideal $\mathrm{in}_w(I)$ is a toric ideal:

$$\mathrm{in}_w(I) = \langle P_{13}P_{24} - P_{14}P_{23} \rangle$$

Matching Fields from Induced Weight Vectors

Question.

Which weight vectors give toric degenerations for the Grassmannian?

- Recall that $I_{k,n} = \ker(\mathbb{C}[P_J] \to \mathbb{C}[x_{i,j}] : P_J \mapsto \det(X_J))$.
- Let $v \in \mathbb{R}^{k \times n}$ be a weight vector for $\mathbb{C}[x_{i,j}]$.
- Induced weight $w \in \mathbb{R}^{\binom{n}{k}-1}$ for $\mathbb{C}[P_J]$ is $w(P_J) = v(\det(X_J))$.

Example: $G_{2,4}$.

Let $v = \begin{bmatrix} 1 & 2 & 0 & 8 \\ 2 & 5 & 2 & 4 \end{bmatrix} \in \mathbb{R}^{2\times 4}$ be a weight for $\mathbb{C}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$.
Induced weight: $w(P_{12}) = v(x_1 y_2 - x_2 y_1) = \min\{1+5, 2+2\} = 4$.

- For any $f \in I_{k,n}$, we have that $\mathrm{in}_w(f)$ is not a monomial.
- A **(coherent) matching field** Λ is *induced* by a vector $v \in \mathbb{R}^{k\times n}$ if $\mathrm{in}_v(\det(X_J))$ is a monomial for each maximal minor $\det(X_J)$.

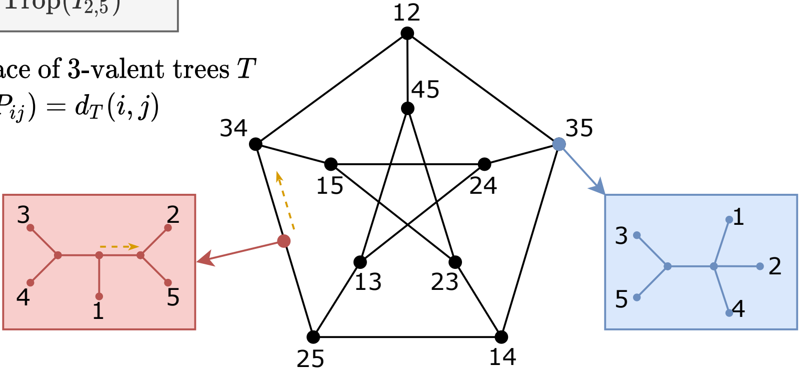
Tropicalisation

- The **tropicalisation** $\text{Trop}(I) \subset \mathbb{R}^n$ is the collection of weight vectors w such that initial ideal $\text{in}_w(I)$ contains no monomials.
- A weight $w \in \text{Trop}(I)$ gives rise to a toric degeneration if $\text{in}_w(I)$ is a **toric ideal**, i.e. it is *binomial* and *prime*.

$\text{Trop}(I_{2,5})$

Space of 3-valent trees T

$w(P_{ij}) = d_T(i,j)$



Block Diagonal Matching Fields

- Mohammadi and Shaw show that not all matching fields (*hexagonal* matching fields) give rise to toric degenerations.
- The **2-block diagonal matching field** B_ℓ, for $\ell \in \{0, \dots, n-1\}$ is the matching field induced by the weight

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \ell & \ell - 1 & \cdots & 1 & n & n-1 & \cdots & \ell + 1 \\ 2n & 2(n-1) & \cdots & \cdots & \cdots & \cdots & \cdots & 2 \\ \vdots & \vdots & & & & & & \vdots \\ n(k-1) & (n-1)(k-1) & \cdots & \cdots & \cdots & \cdots & \cdots & k-1 \end{bmatrix}.$$

Theorem (C-Mohammadi 2020).

Each matching field B_ℓ produces a toric degeneration of $G_{k,n}$.

- The case B_0 is the **Gelfand-Tsetlin degeneration**.

Toric Ideals from Matching Fields

- Suppose $v \in \mathbb{R}^{k \times n}$ induces a matching field Λ, i.e. $\mathrm{in}_v(\det(X_J))$ is a monomial for each J. Let $w \in \mathbb{R}^{\binom{n}{k}-1}$ be the induced weight vector.
- Let $J_\Lambda = \ker(\psi_\Lambda)$ where $\psi_\Lambda : \mathbb{C}[P_J] \to \mathbb{C}[x_{i,j}]$ is the monomial map sending P_J to $\mathrm{in}_v(\det(X_J))$.
- If Λ gives produces a toric degeneration of $G_{k,n}$, then the ideal of the toric variety is exactly $\mathrm{in}_w(I_{k,n}) = J_\Lambda$.
- Moreover, the polytope P_Λ of the toric variety is the convex hull of the exponent vectors of $\psi_\Lambda(P_J)$ (**matching field polytope**).

Example: $G_{2,4}$.

The matching field B_2 is induced by the weight vector $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

Since $\psi_{B_2}(P_{12}) = x_1 y_2$, we get the vertex $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ of P_{B_2}.

The f-vector of P_Λ is $(6, 13, 13, 6, 1)$.

3. *Our Results*

Toric Degenerations of the Grassmannian

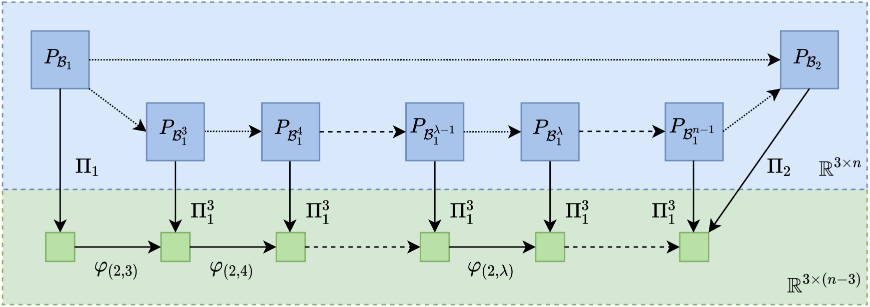
- The GT-polytope is the polytope of the matching field B_0.

Theorem (C-Higashitani-Mohammadi 2021).

- Any pair of 2-block diagonal matching field polytopes for the Grassmannian $G_{k,n}$ are mutation equivalent.
- There exists a sequence of mutations which passes only through matching field polytopes.
- If the polytope of a matching field Λ is mutation equivalent to the GT-polytope, then Λ gives a toric degeneration of the Grassmannian.

- As a result we extend the known family of toric degenerations for $G_{k,n}$.
- For each tropical map we construct, the *factor polytope F* is a line segment.
- We show that tropical maps also preserve the *integer decomposition property (IDP)* for this family of polytopes.

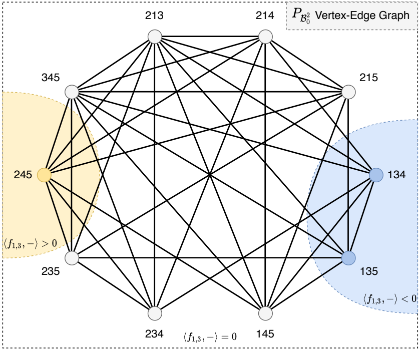
Mutation Diagram

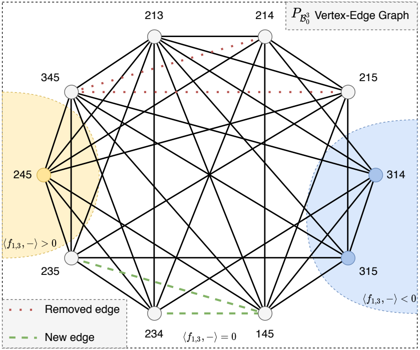


Mutation from the block diagonal matching field polytope $P_{\mathcal{B}_1}$ to $P_{\mathcal{B}_2}$.

- Blue boxes are matching field polytopes.
- Π_i^j are linear maps acting as isomophisms on the polytopes.
- $\varphi_{(i,j)}$ are tropical maps.

Vertex-Edge Graph under Mutation





Vertex-Edge graph of matching field polytopes for Gr(3, 5) *which differ by a single mutation.*

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