Fano spherical varieties of small dimension and rank

with P.L. Montagard

1 Motivation

Classification of low dimensional Fano mfds

→ Fano 3folds [Iskovskikh, Mori Mukai] 105 deformation classes

→ 4 folds hard pbm

\+ group action

→ reductive cplx grp action

→ Fano 3folds with ∞ group action [Cheltsov, Przyjalkowski, Shramov 2019]

→ toric Fano 4 folds [Batyrev 1999, Sato 2000] 124 different

why grp actions?

* mfds with symmetries are easier to classify

  e.g. toric $\longleftrightarrow$ combinatorial data

* (X, D) pairs

  Fano: X; divisor: D

symmetries $\mathrm{Aut}(X,D) \subset \mathrm{Aut}(X)$

usually strict

② Main result

Definition: * X normal alg variety

$\circlearrowleft$

G complex connected reductive group

is <u>spherical</u> if $\forall B \subset G$ Borel subgroup, B acts on X with an open orbit.

* $G \curvearrowright X$, B fixed Borel $\subset G$

<u>weight lattice</u> $M :=$ set of B-weights of B-eigenvectors in $B \curvearrowright \underbrace{\mathbb{C}(X)}_{\text{rational functions on } X}$

$\cap$

$X^*(B)$ group of characters of B.

* <u>rank</u> $:=$ rk(M) as a free abelian group

$M \simeq \mathbb{Z}^{\mathrm{rk}}$ $(0 \leqslant \mathrm{rk} \leqslant \dim X)$*

* X is Fano if K_X^{-1} is ample
* X is locally factorial if any Weil divisor on X is Cartier.

Theorem [D.–Montagard]: Classification (up to G-equivariant isom) of $X \circlearrowleft G$ w/

X loc. fact. Fano, $\dim X \leq 4$

$X \circlearrowleft G$ faithful & spherical of rank ≤ 2.

+ associated combinatorial data (analogous to toric: "colored fans")

+ Picard number, anticanonical degree, K-stability

rk \ dim	1	2	3	4	
homogeneous [0	1	2	6	9	260
1	1	5	13	57	
2	0	5	44	194	
3	0	0	18	?	
4	0	0	0	124	

337

toric

https://shorturl.at/hsW18

Comments: * dim = 4, rk = 3

WIP by Gertrude Hamm

expect hundreds of examples

* dim 1: $\mathbb{P}^1 \circlearrowleft SL_2$ homogenous

$\mathbb{P}^1 \circlearrowleft \mathbb{C}^*$ toric

* dim 2: "well known".

* dim 3: follows from PhD theses of:

Pasquier 2006 — in French

Hofscheier 2015 — not available online

unpublished.

* Caveat : underlying X not easy to identify
not done completely in our result.

But : from computations of geometric data :

at least 117 $\neq$ underlying X

— 42 $\neq$ non toric underlying X among 4folds

— 93 not KE

— 24 KE

* smoothness ?

if underlying X toric then LG fact $\Leftrightarrow$ smooth
toroidal

at least 321 out of 337 are smooth
3 not smooth
13 unknown

(3) Sketch of proof

$\boxed{A}$ $X \circlearrowleft G$ spherical $\qquad \exists$ open G-orbit $G/H \subset X$

↳ classify these possible spherical homogeneous spaces G/H

with $\dim \leq 4$ and $\underline{rk \leq 2}$.

$\boxed{B}$ Theory of spherical embeddings:

Fix G/H

G-equiv embeddings $G/H \subset X$ $\longleftrightarrow$ colored fans

(4) Local structure theorem

Standing assumption: $G = G^{sc} \times (\mathbb{C}^*)^n$ st $G^{sc} \curvearrowright X$ with finite conn'd kernel

G^{sc}: semisimple, simply connected

$(\mathbb{C}^*)^n \curvearrowright X$ faithful

Theorem [Brion Luna Vust 1986]: Assume BH/H is open in G/H.

Let $P := \mathrm{Stab}(BH/H) \subset G$ "adapted" parabolic subgroup.

$\exists$ Levi decomposition $P = LP^u$ st:

(i) $P \cap H = L \cap H \supset [L, L]$

and if C is connected center of L, then

(ii) $P^u \times C/C\cap H \longrightarrow BH/H$ isomorphism.

$(p, x) \longmapsto p \cdot x$

Consequences: (ii) $\Rightarrow \dim G/H = \dim BH/H$

$$= \dim(P^u) + \dim(C/C\cap H)$$

$$= \dim(G/P) + \mathrm{rk}(X \circlearrowleft G)$$

(i) + standing assumption $\Rightarrow$ P does not contain a simple factor of G.

Rem: $G/P = G^{sc}/P\cap G^{sc}$.

$\rightarrow$ strong restrictions on possible G^{sc}

If $\dim(G/H) = 4$

$\mathrm{rk}\,0 \Rightarrow G/H = G/P$ projective homogeneous

$\mathrm{rk}\,1 \Rightarrow \dim G/P = 3$

$\mathrm{rk}\,2 \Rightarrow \dim G/P = 2$

$\hookrightarrow G/P \in \{ \mathbb{P}^2 \circlearrowleft SL_3 , \mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft SL_2^2 \}$

$\Rightarrow G^{sc} \in \{ SL_3, SL_2^2 \}$

$\mathrm{rk}\,3 \Rightarrow G^{sc} = SL_2$

$\mathrm{rk}\,4 \Rightarrow G = (\mathbb{C}^+)^4$

(5) Parabolic induction

Defn: G/H is obtained by parabolic induction if

$$G = \frac{G \times G_0/H_0}{Q}$$

where: Q proper parabolic subgp of G

$\pi : Q \longrightarrow G_0$ reductive quotient

$Q \curvearrowright G \times G_0/H_0$ by $q \cdot (g, x) = (g q^{-1}, \pi(q) \cdot x)$

Key properties: * H spherical $\Leftrightarrow$ H_0 spherical

* detected at Lie alg level: $\mathfrak{q}^u \subset \mathfrak{h} \subset \mathfrak{q}$

* rk 1 spherical are classified up to parabolic induction.
[Akhiezer]

H

⑥ Rk 2

[Douglas, Repka 2006]: explicit classification of Lie subalgebras of $\underbrace{\mathfrak{sl}_3}_{\mathfrak{g}^{sc}}$, $\underline{\mathfrak{sl}_2 \oplus \mathfrak{sl}_2}$ up to conjugation.

Upshot: most are obtained by parabolic induction

Not finished yet: * throw in the torus factor
* classify $_{\wedge}$ parabolic inductions
possible

⑦ Fano embeddings

[Brion 1989] → combinatorial description of Picard grp → Picard number

deg of a line bundle,

$$\overset{\cup}{K_X^{-1}}$$

[Brion 1997] → K_X^{-1}

[Gagliardi - Hofscheier 2015] polytope interpretation

[D. 2020] combinatorial criterion for K-stability of spherical Fano varieties.

$\exists$ smoothness criterion for spherical varieties

smooth

$\cap$

loc. factorial

$\cap$

terminal