Nottingham - Algebraic geometry seminar

Stability of toric vector bundles

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Introduction

Let X be a smooth complete toric variety of dimension d.

algebraic closure of a "torus" $T = (\mathbb{C}^*)^d$

($T \subset X$ as an open subset + $T \curvearrowright T$ extends to an action $T \curvearrowright X$)

Σ fan = set of cones τ in $\mathbb{R}^d$ stable under intersection and taking faces.

$x, y \in \tau$, $\lambda \in \mathbb{R}_{\geq 0}$ $\Rightarrow x + \lambda y \in \tau$

We denote by $\Sigma(k)$ the subset of Σ formed by the cones of dimension k.

(varieties of codimension k added "at infinity" to the torus)

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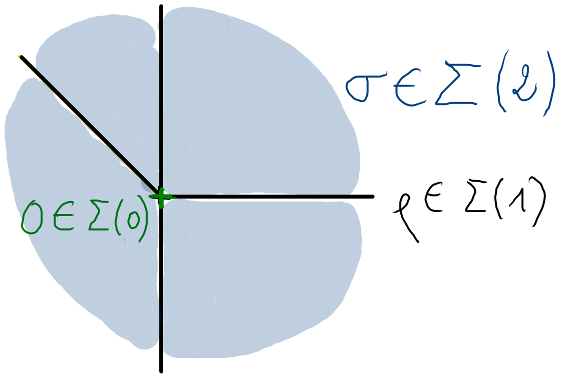
Σ fan



with
- $\mathrm{Supp}(\Sigma) = \mathbb{R}^d$
- each cone is generated by a subset of a basis of $\mathbb{R}^d$.

tvb on X := vector bundle $\mathcal{E}$ on X (ie a locally free $\mathcal{O}_X$-module of finite rank) with a T-action compatible with the T-action on X

→ π is T-equivariant

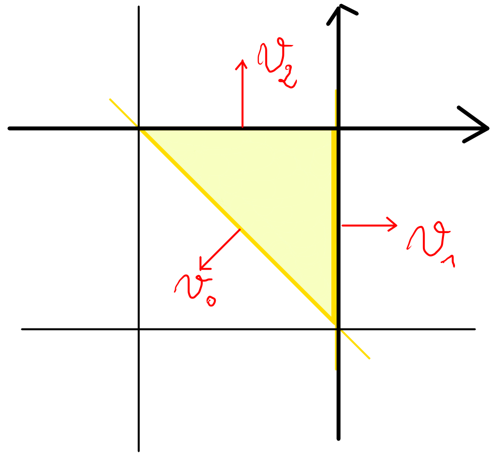
→ T acts linearly on the fibers

$$\begin{array}{c} \mathrm{Spec}(\mathrm{Sym}\,\mathcal{E}) \\ \downarrow \pi \\ X \end{array}$$

Introduction

a toric line bundle $\mathcal{O}_X(D)$ $\rightsquigarrow$ Newton polytope P_D
(or a toric divisor D)

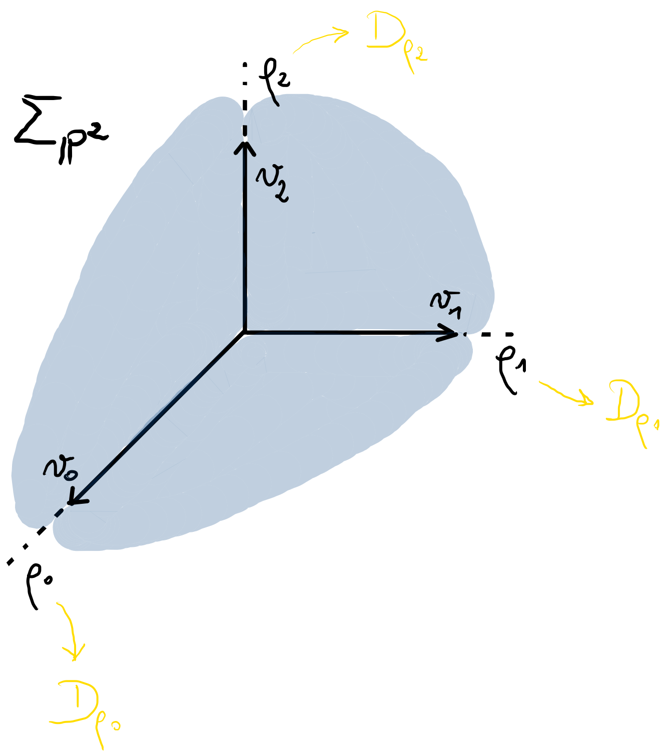
$\mathcal{O}_{\mathbb{P}^2}(1D_{\rho_0}+0D_{\rho_1}+0D_{\rho_2})$ $\rightsquigarrow$



$\rightsquigarrow$

- global sections of D
- positivity of D : big, ample, ...

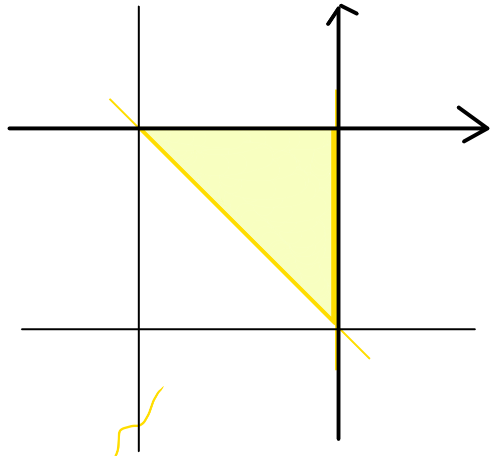
...



Introduction

a toric line bundle $\mathcal{O}_X(D)$ $\rightsquigarrow$ Newton polytope P_D
(or a toric divisor D)

$\mathcal{O}_{\mathbb{P}^2}(D_{\rho_0})$ $\rightsquigarrow$



$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ is 3-dimensional

$\mathcal{O}_{\mathbb{P}^2}(1)$ is big, ample

- global sections of D
- positivity of D : big, ample, ...

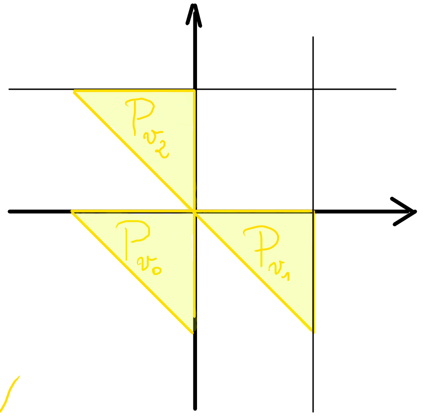
...

Introduction

Di Rocco–Jabbusch–Smith
2014

a toric vector bundle $\mathcal{E}$ $\rightsquigarrow$ parliament of polytopes $PP_{\mathcal{E}}$

$\mathcal{T}_{\mathbb{P}^2}$ $\rightsquigarrow$



$H^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2})$ is 8-dimensional

$\mathcal{T}_{\mathbb{P}^2}$ is big, ample, globally generated

- global sections of $\mathcal{E}$
- positivity of $\mathcal{E}$: big, ample, ..., globally generated
- stability ?

Introduction

why stability is important ?

Introduction

Stable vector bundles and moduli spaces

A moduli space is a geometric space whose points represent algebro-geometric objects of some kind.

eg. $\mathbb{P}^n$ is a moduli space which parametrizes the lines in $\mathbb{C}^{n+1}$ passing through the origin.

a moduli space of vector bundles?

rather a moduli space of stable vector bundles.

Introduction

Stable vector bundles, as elementary bricks of vector bundles

Harder-Narasimhan filtration :

Let $\mathcal{E}$ be a vector bundle over a smooth projective ~~curve~~ variety X.
There exists a unique filtration by subbundles $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_m = \mathcal{E}$
s.t. • $\forall i \in \{1, \dots, m\}$, $\mathcal{E}_i / \mathcal{E}_{i-1}$ is a semistable ~~vector bundle~~ coherent sheaf (of slope λ_i)
• $\lambda_1 > \lambda_2 > \dots > \lambda_m$

$\mathcal{M}$: if $\dim X > 1$.

notions of Stability

- for vector bundles $\mathcal{E}$ on smooth projective curves X: "$\mathcal{E}$ is stable iff all its subbundles are less-ample
  ie $\forall \mathcal{F} \subset \mathcal{E}$ proper subbundle, $\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{rk(\mathcal{F})} < \mu(\mathcal{E})$"
  [Mumford, 1963]

- in higher dimension:
  $\dim X = n$

slope-stability

$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{rk(\mathcal{F})}$

polarization of X
(ample divisor on X)

[Takemoto, 1972]

Gieseker-stability

based on all Chern classes

[Gieseker, 1977]

Bridgeland-stability

doesn't rely on a polarization anymore

[Bridgeland, 2007]

Slope-stability of toric v.b.

Fix a polarized toric variety (X, H). The slope of a coherent sheaf $\mathcal{F}$ is $\boxed{\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F})\cdot H}{rk(\mathcal{F})}}$.

A toric vector bundle $\mathcal{E}$ is slope-stable

iff every proper subsheaf $\mathcal{F}$ satisfy $\mu_H(\mathcal{F}) < \mu_H(\mathcal{E})$

iff every proper equivariant saturated subsheaf $\mathcal{F}$ satisfy $\mu_H(\mathcal{F}) < \mu_H(\mathcal{E})$. [Kool, 11]

Plan

1st Step

reflexive equivariant sheaf $\mathcal{F}$ $\rightsquigarrow$ $PP_{\mathcal{F}}$ parliament of polytopes $\rightsquigarrow$ $P_{\mathcal{F}}$ average polytope $\rightsquigarrow$ $\mu_H(\mathcal{F})$ (for any H) slope

eg. an equivariant saturated subsheaf of $\mathcal{E}$

slope from the parliament of polytopes

Plan

reflexive equivariant sheaf

$\mathcal{F} \rightsquigarrow PP_{\mathcal{F}} \rightsquigarrow P_{\mathcal{F}} \rightsquigarrow \mu_H(\mathcal{F})$ (for any H)

parliament of polytopes

average polytope

slope

eg. an equivariant saturated subsheaf of $\mathcal{E}$

2nd Step

If $\mathcal{F}$ is an equivariant saturated subsheaf of $\mathcal{E}$ then $PP_{\mathcal{E}}$ and $PP_{\mathcal{F}}$ may be related

Plan

2nd Step

For any tvb $\mathcal{E}$,

a family of equivariant saturated subsheaves $\mathcal{F}$ of $\mathcal{E}$

(sufficient to test the stability of $\mathcal{E}$)

and their parliaments $PP_{\mathcal{F}}$

can be read on $PP_{\mathcal{E}}$.

1^{st} Step :

get the slope from the parliament of polytopes

Parliaments of polytopes

Di Rocco–Jabbusch–Smith 2014

a equivariant reflexive sheaf $\mathcal{F}$ $\rightsquigarrow$ Parliament of polytopes $PP_{\mathcal{F}}$

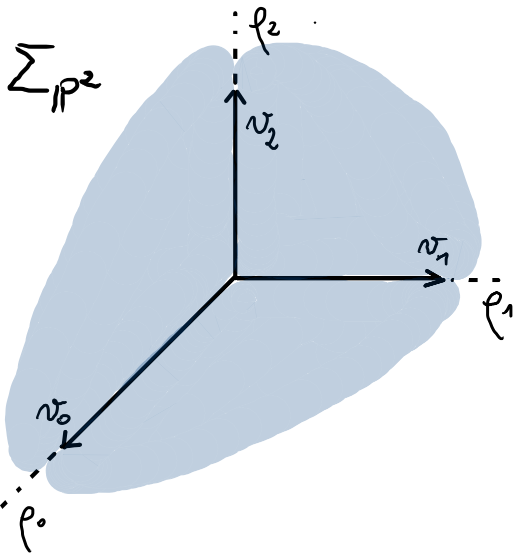
$=$ visual representation of Klyachko classification

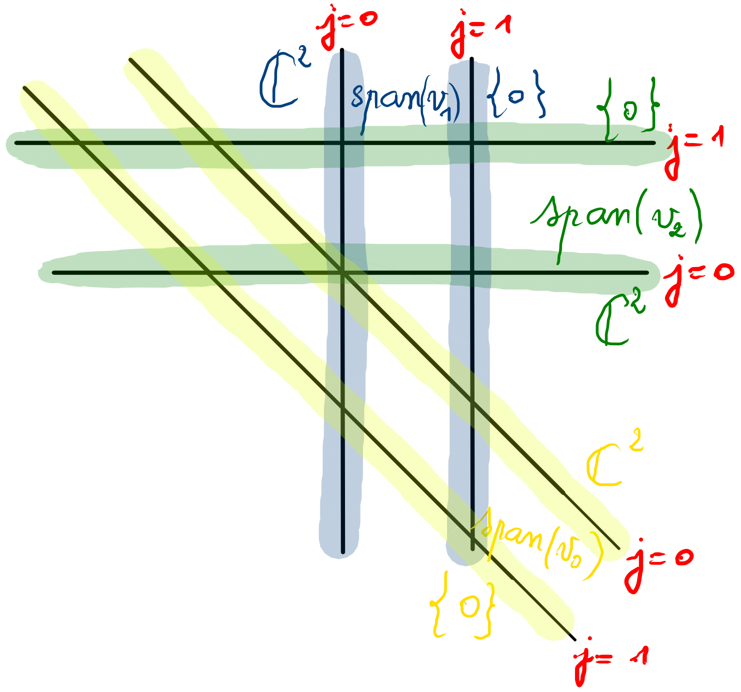
<u>particular case:</u> a toric vector bundle $\mathcal{E}$

a $\mathbb{Z}$-filtration of $\mathbb{C}^{rk(\mathcal{F})}$ for each ray $\rho_i \in \Sigma(1)$

$+$ compatible with each other

e.g. tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on $\mathbb{P}^2$





$$\begin{array}{ll}
 & \overbrace{\qquad\qquad\qquad\qquad}^{j\le 0} \quad \overbrace{\qquad\qquad}^{0<j\le 1} \quad \overbrace{\qquad\qquad\qquad}^{1<j} \\
\rho_0: & \cdots \supseteq \mathbb{C}^2 \supseteq \cdots \supseteq \mathbb{C}^2 \supseteq \mathrm{span}(v_0) \supseteq \{0\} \supseteq \cdots \supseteq \{0\} \supseteq \cdots \\
\rho_1: & \cdots \supseteq \mathbb{C}^2 \supseteq \cdots \supseteq \mathbb{C}^2 \supseteq \mathrm{span}(v_1) \supseteq \{0\} \supseteq \cdots \supseteq \{0\} \supseteq \cdots \\
\rho_2: & \cdots \supseteq \mathbb{C}^2 \supseteq \cdots \supseteq \mathbb{C}^2 \supseteq \mathrm{span}(v_2) \supseteq \{0\} \supseteq \cdots \supseteq \{0\} \supseteq \cdots \\
 & \xrightarrow{\quad j\in\mathbb{Z}\quad} \qquad j=0 \qquad\qquad j=1 \qquad\qquad j=2
\end{array}$$

Parliaments of polytopes

Di Rocco–Jabbusch–Smith 2014

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= visual representation of Klyachko classification

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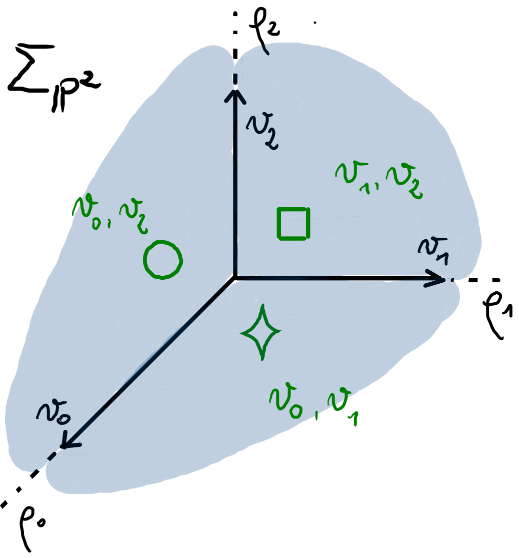
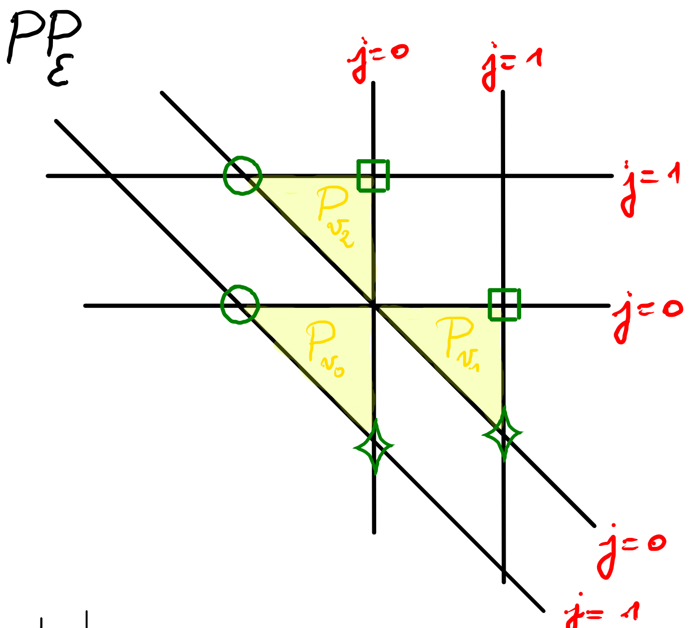
+ compatible with each other





$j \leq 0$ | $0 < j \leq 1$ | $1 < j$

$$\begin{array}{lcccccccc}
\rho_0: & \dots \supseteq \mathbb{C}^2 \supseteq \dots \supseteq \mathbb{C}^2 & \supseteq & \mathrm{span}(v_0) & \supseteq & \{0\} \supseteq \dots \supseteq \{0\} \supseteq \dots \\
\rho_1: & \dots \supseteq \mathbb{C}^2 \supseteq \dots \supseteq \mathbb{C}^2 & \supseteq & \mathrm{span}(v_1) & \supseteq & \{0\} \supseteq \dots \supseteq \{0\} \supseteq \dots \\
\rho_2: & \dots \supseteq \mathbb{C}^2 \supseteq \dots \supseteq \mathbb{C}^2 & \supseteq & \mathrm{span}(v_2) & \supseteq & \{0\} \supseteq \dots \supseteq \{0\} \supseteq \dots
\end{array}$$

$\xrightarrow{j \in \mathbb{Z}}$ $j=0$ $j=1$ $j=2$

$$P_e = \bigcap_{\rho_i \in \Sigma(1)} \left\{ m \in \mathbb{R}^d \;\middle|\; \langle m, v_i \rangle \leq \max\{ j \mid e \in E^i(j) \} \right\}$$

Parliaments of polytopes

Di Rocco – Jabbusch – Smith 2014

a equivariant reflexive sheaf $\mathcal{F}$ $\rightsquigarrow$ Parliament of polytopes $PP_{\mathcal{F}}$

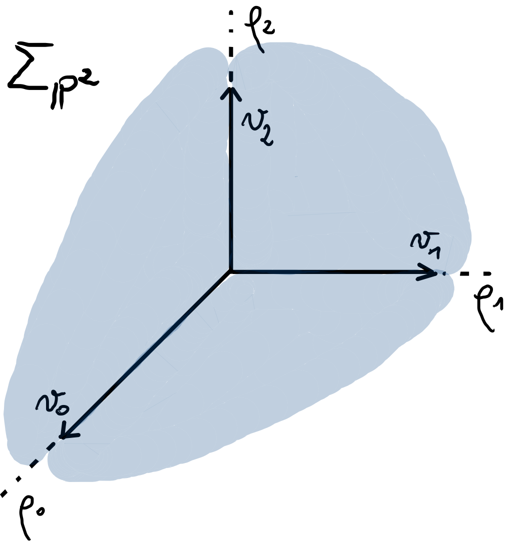
particular case: a toric vector bundle $\mathcal{E}$

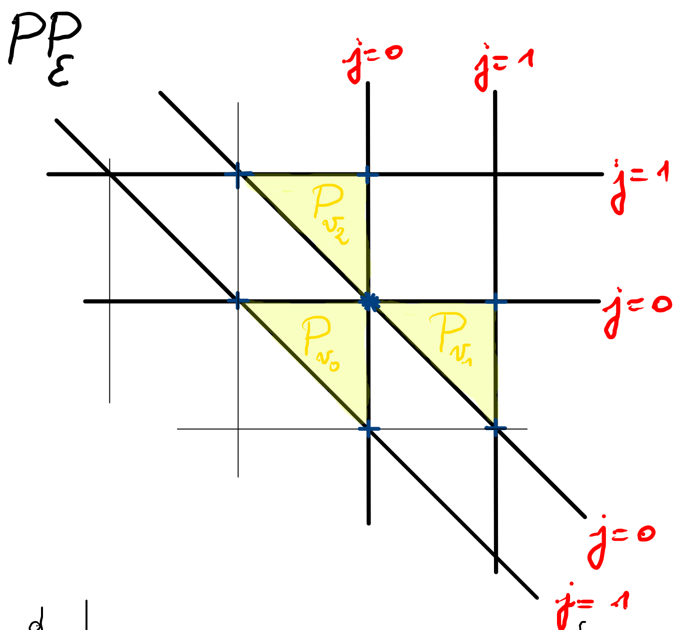
= a set of indexed polytopes defined by r hyperplanes in each directions $\vec{v}_i$

s.t. any point $u \in P_e \cap \mathbb{Z}^d$ corresponds to a global section $s = e \otimes \chi^{-u}$

generates the set of global sections

e.g. tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on $\mathbb{P}^2$

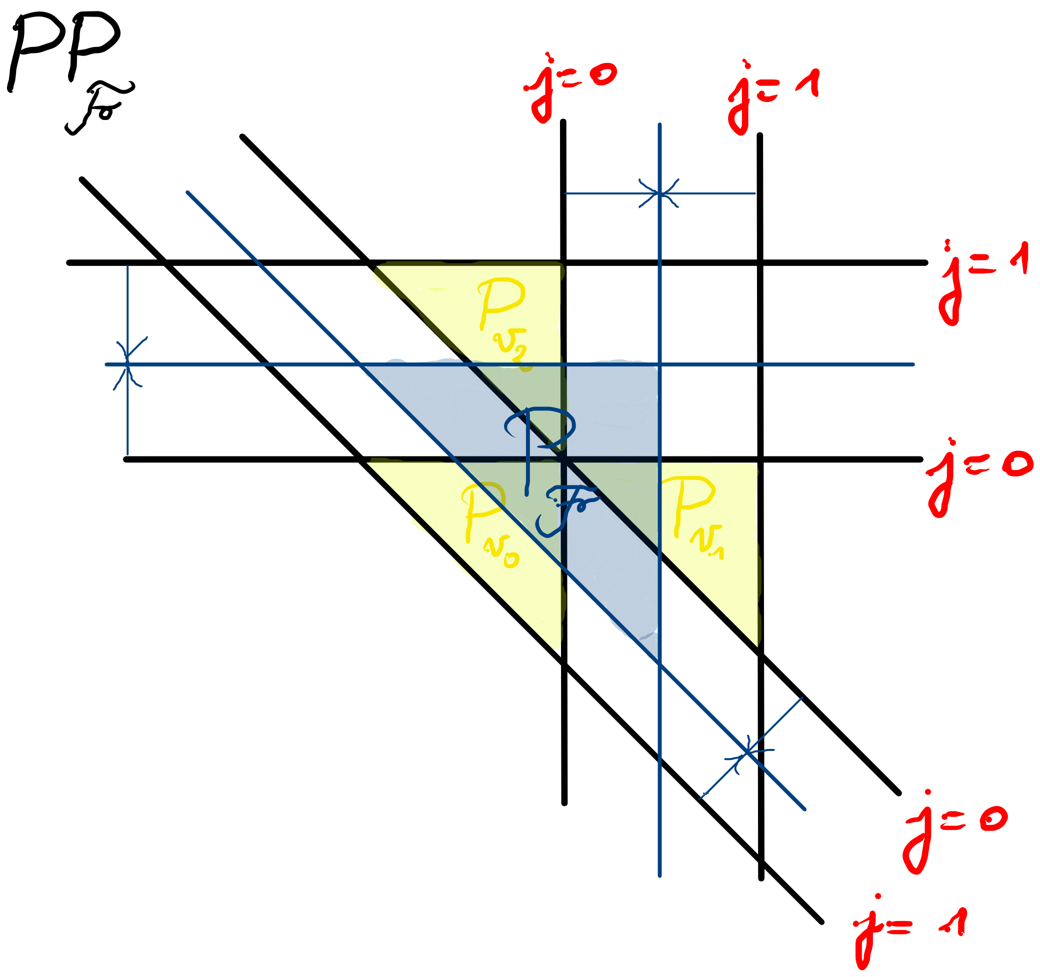




$$P_e = \bigcap_{\rho_i \in \Sigma(1)} \left\{ m \in \mathbb{R}^d \;\middle|\; \langle m, v_i \rangle \leq \max \{ j \mid e \in E^i(j) \} \right\}$$

Stability of toric vector bundles

How to construct the average polytope $P_{\mathcal{F}}$?

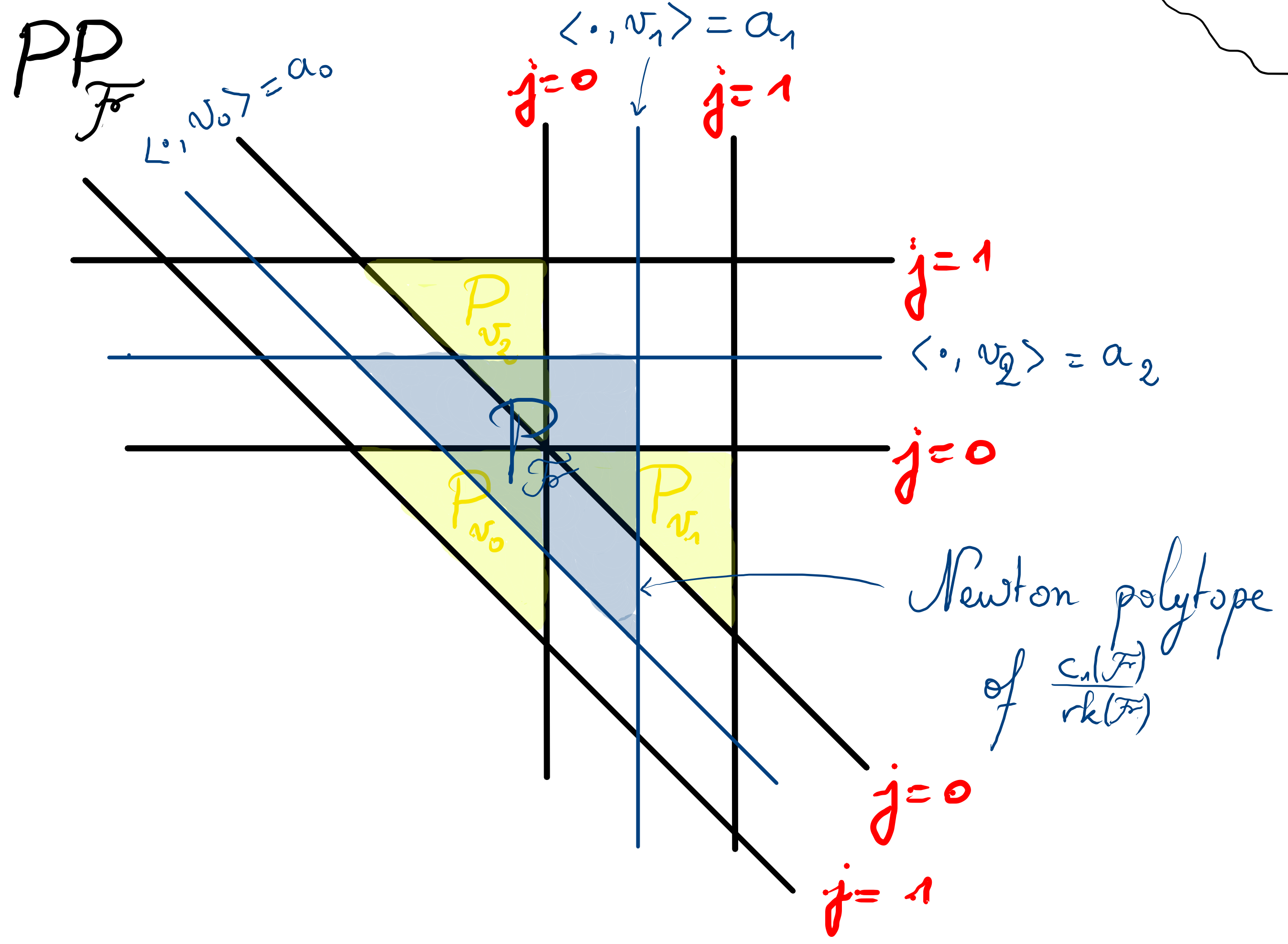


[Payne, '05] ⇝ formula for chern classes

I defined the average polytope as being the Newton polytope of $\frac{c_1(\mathcal{E})}{rk(\mathcal{E})}$.

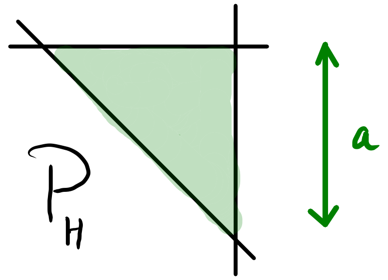
Stability of toric vector bundles

How to obtain the slope $\mu_H(\mathcal{F})$ from the average polytope $P_{\mathcal{F}}$?



H → polarization of X.

The Newton polytope of H is



We define some numbers

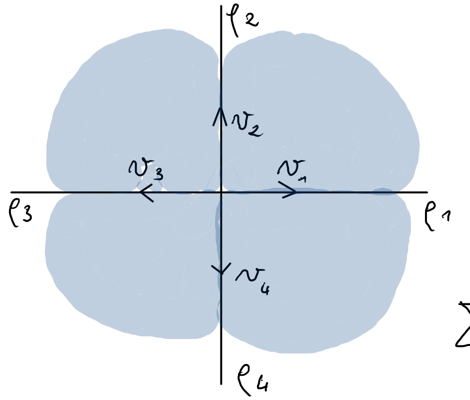
$$t_i = vol(P_{H,i}) \times \frac{(n-1)!}{\| v_i \|}$$

$$t_0 = t_1 = t_2 = a$$

$$\boxed{\mu_H(\mathcal{F}) = \frac{H^{n-1} \cdot c_1(\mathcal{F})}{rk(\mathcal{F})} = \sum_{\rho_i \in \Sigma(1)} t_i \, a_i}$$

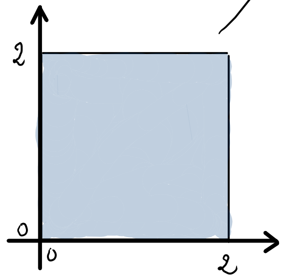
Remark: In the case $X = \mathbb{P}^2$, $\mu_H(\mathcal{F}) < \mu_H(\mathcal{F}') \iff P_{\mathcal{F}}$ is bigger than $P_{\mathcal{F}'}$.

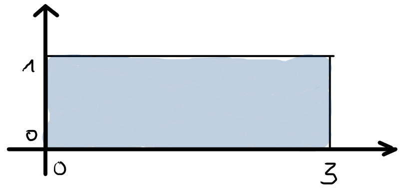
e.g. $X = \mathbb{P}^1 \times \mathbb{P}^1$

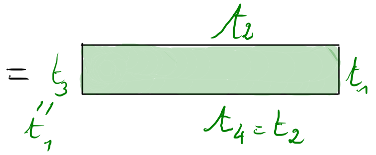


$\Sigma(\mathbb{P}^1 \times \mathbb{P}^1)$

We would like to compare the slopes of $\mathcal{F}$ and $\mathcal{F}'$ by means of their average polytopes:

$P_{\mathcal{F}} =$ 

and $P_{\mathcal{F}'} =$ 

if $P_H =$ 

then $\mu_H(\mathcal{F}) = 2t_1 + 2t_2 + 0t_3 + 0t_4$ and $\mu_H(\mathcal{F}') = 3t_1 + t_2 + 0t_3 + 0t_4$.

1st case: $t_2 > t_1$ $\qquad \mu_H(\mathcal{F}) > \mu_H(\mathcal{F}')$

2nd case: $t_2 = t_1$ $\qquad \mu_H(\mathcal{F}) = \mu_H(\mathcal{F}')$

3rd case: $t_2 < t_1$ $\qquad \mu_H(\mathcal{F}) < \mu_H(\mathcal{F}')$

2^{nd} Step :

For any tvb $\mathcal{E}$,

a family of equivariant saturated subsheaves $\mathcal{F}$ of $\mathcal{E}$

(sufficient to test the stability of $\mathcal{E}$)

and their parliaments $PP_{\mathcal{F}}$

can be read on $PP_{\mathcal{E}}$.

Parliament of equivariant saturated subsheaves

Let $\mathcal{E}$ be a toric <u>vector bundle</u>.

Klyachko classification $\longleftrightarrow$ a $\mathbb{Z}$-filtration $(E^i(j))_{j\in\mathbb{Z}}$ of $\mathbb{C}^{rk(\mathcal{E})}$ for each ray $\rho_i \in \Sigma(1)$

\+ compatible with each other

$$\text{equivariant saturated subsheaves of } \mathcal{E} \xleftrightarrow[\text{classification}]{\text{Klyachko}} (E^i(j) \cap F)_{j\in\mathbb{Z}}, \text{ with } F \text{ subspace of } \mathbb{C}^{rk(\mathcal{E})}$$

(as reflexive equivariant sheaves)

Dasgupta – Dey – Khan, 19

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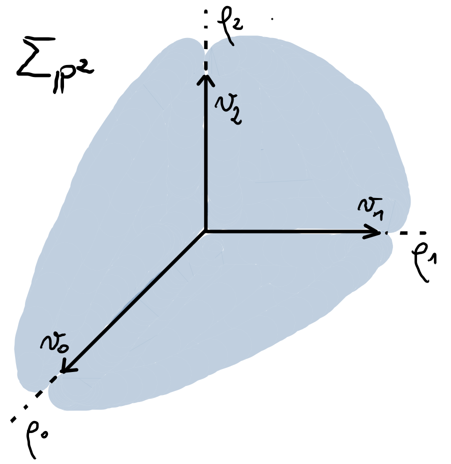

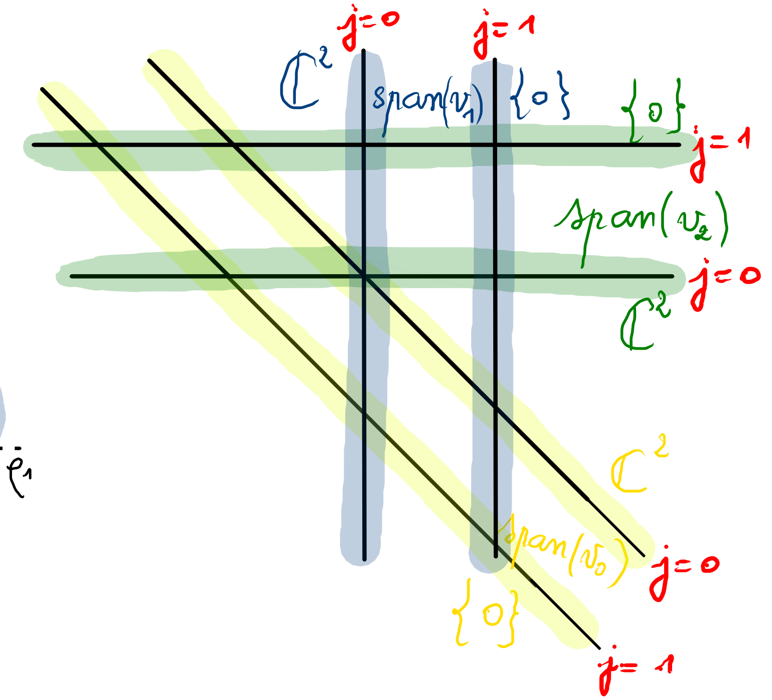

	$j \leq 0$	$0 < j \leq 1$	$1 < j$
ρ_0:	$\cdots \supseteq \mathbb{C}^2 \supseteq \cdots \supseteq \mathbb{C}^2 \supseteq$	$\operatorname{span}(v_0) \supseteq$	$\{0\} \supseteq \cdots \supseteq \{0\} \supseteq \cdots$
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ρ_2:	$\cdots \supseteq \mathbb{C}^2 \supseteq \cdots \supseteq \mathbb{C}^2 \supseteq$	$\operatorname{span}(v_2) \supseteq$	$\{0\} \supseteq \cdots \supseteq \{0\} \supseteq \cdots$

Parliament of equivariant saturated subsheaves

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equivariant saturated subsheaves of $\mathcal{E}$ $\xleftrightarrow[\text{classification}]{\text{Klyachko}}$ $(E^i(j) \cap F)_{j\in\mathbb{Z}}$, with F subspace of $\mathbb{C}^{rk(\mathcal{E})}$

e.g. tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on $\mathbb{P}^2$

What are the proper equivariant saturated subsheaves $\mathcal{F}$ of $\mathcal{E}$?

$\hookrightarrow$ they correspond to subspaces F of $\mathbb{C}^2$ of dim 1.

Parliament of equivariant saturated subsheaves

Let $\mathcal{E}$ be a toric vector bundle.

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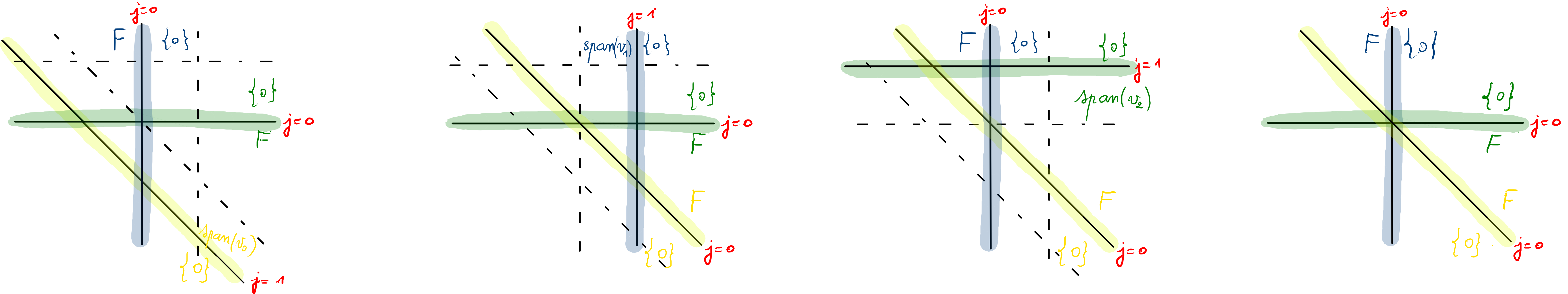
e.g. tangent bundle $\mathcal{E} = \mathcal{T}_{\mathbb{P}^2}$ on $\mathbb{P}^2$

if $F = \mathrm{span}(v_0)$: if $F = \mathrm{span}(v_1)$: if $F = \mathrm{span}(v_2)$: if $v_0, v_1, v_2 \notin F$:



Parliament of equivariant saturated subsheaves

Let $\mathcal{E}$ be a toric vector bundle.

Klyachko classification $\longrightarrow$ a $\mathbb{Z}$-filtration $(E^i(j))_{j\in\mathbb{Z}}$ of $\mathbb{C}^{rk(\mathcal{E})}$ for each ray $\rho_i \in \Sigma(1)$

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equivariant saturated subsheaves of $\mathcal{E}$ $\xleftrightarrow[\text{classification}]{\text{Klyachko}}$ $(E^i(j)\cap F)_{j\in\mathbb{Z}}$, with F subspace of $\mathbb{C}^{rk(\mathcal{E})}$

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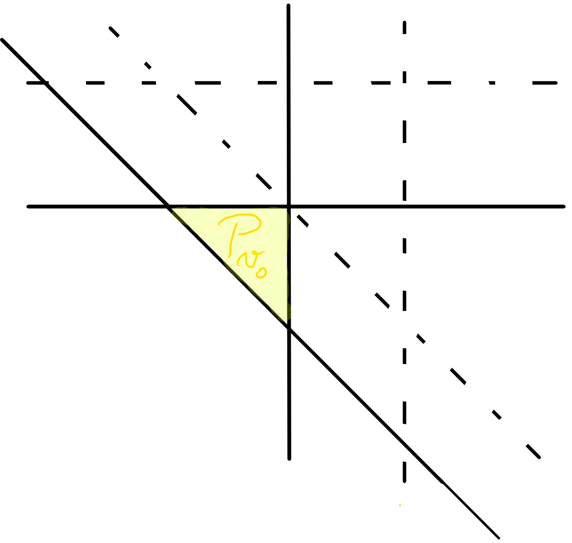
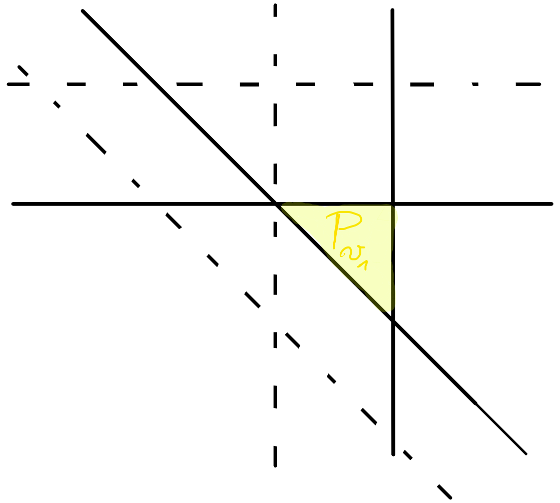
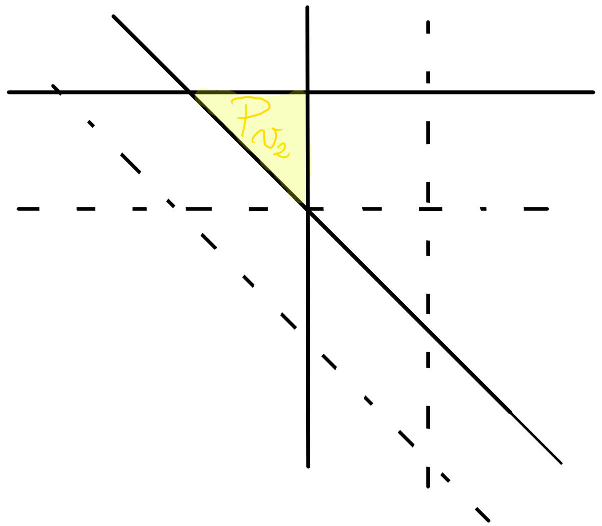


if $F = \mathrm{span}(v_1)$:

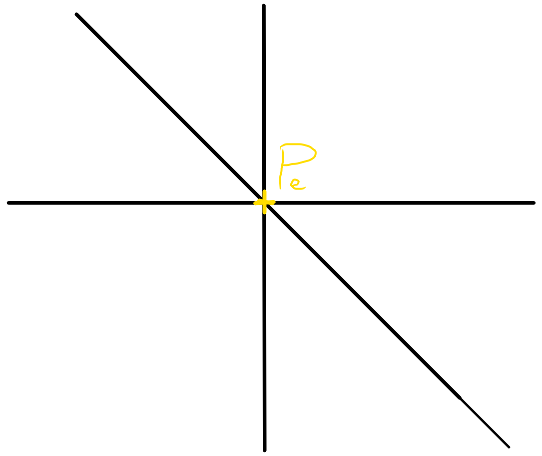


if $F = \mathrm{span}(v_2)$:



if $v_0, v_1, v_2 \notin F$:

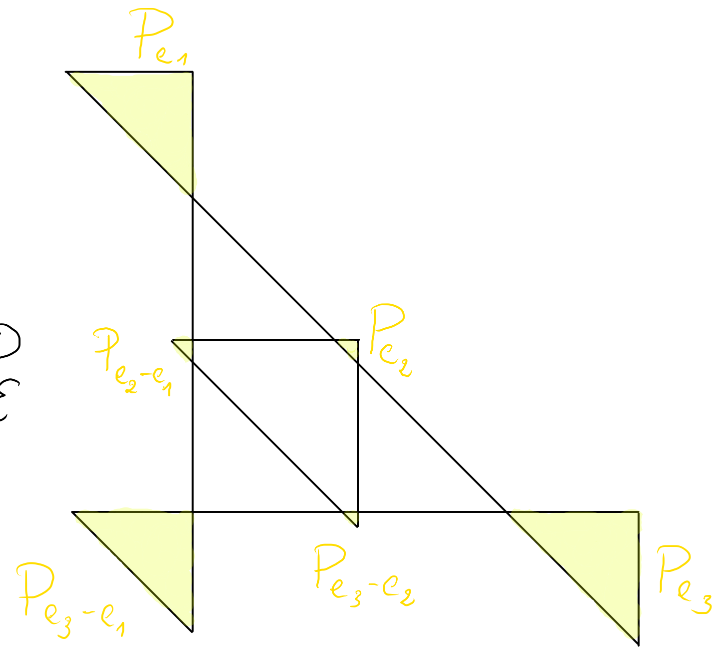
$F = \mathrm{span}(e)$



Stability of toric vector bundles

Another example.

$P\mathcal{P}_{\mathcal{E}}$



Stability of toric vector bundles

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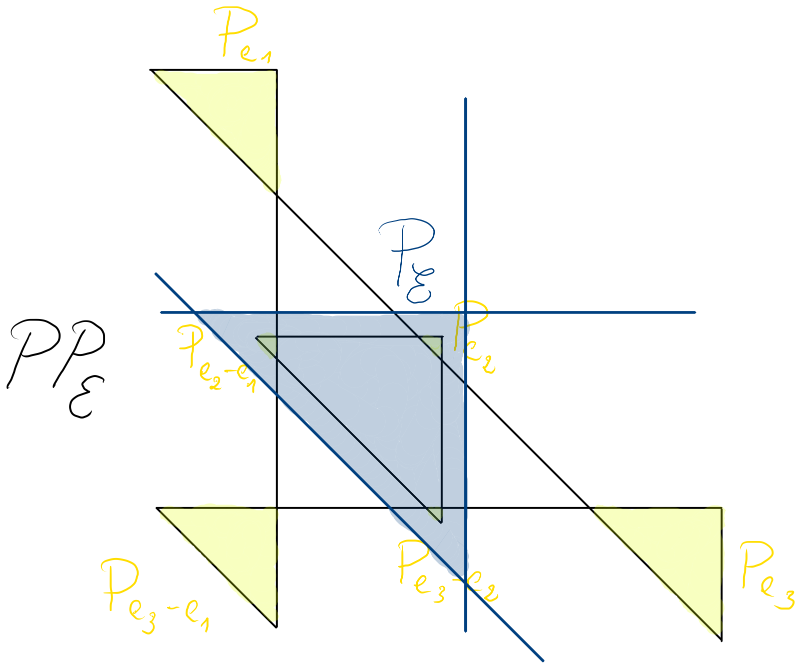
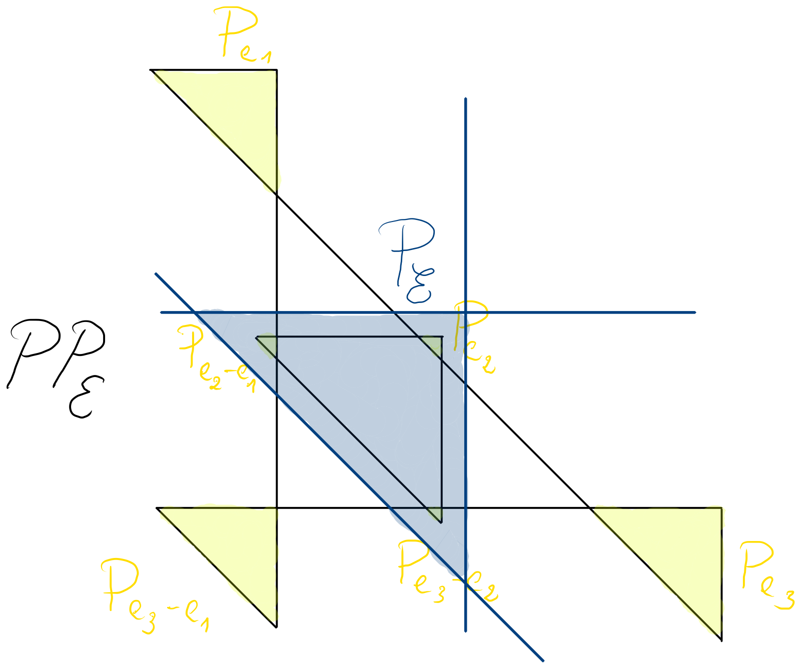


We need to compare $P_{\mathcal{E}}$ with the average polytopes $P_{\mathcal{F}_0}$ of equiv sat. subsheaves $\mathcal{F}$ $\longleftrightarrow$ F generated by indices in $PP_{\mathcal{E}}$

Stability of toric vector bundles

Another example.

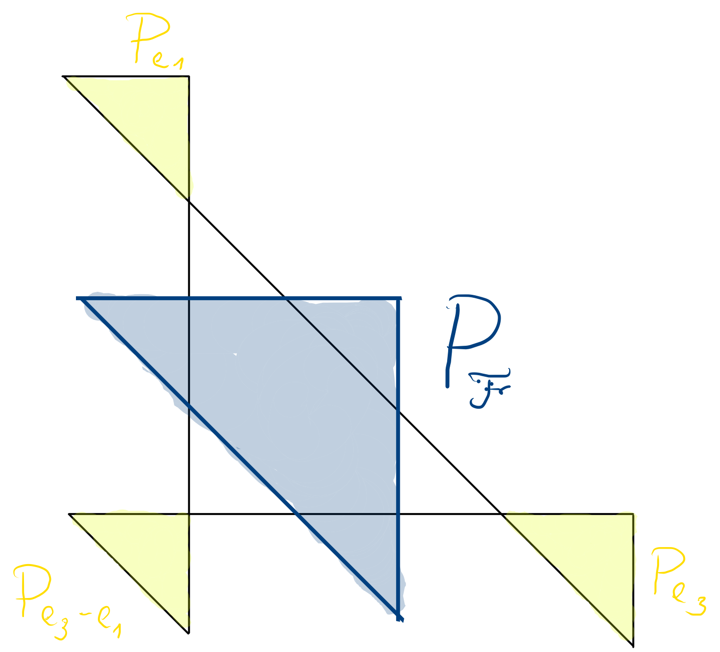


We need to compare $P_{\mathcal{E}}$ with the average polytopes $P_{\mathcal{F}^o}$ of equiv sat. subsheaves $\mathcal{F}$ $\longleftrightarrow$ F generated by indices in $PP_{\mathcal{E}}$

if $\dim F = 1$: $\quad \mu_H(\mathcal{F}) < \mu_H(\mathcal{E})$

Stability of toric vector bundles

Another example.



We need to compare $P_{\mathcal{E}}$ with the average polytopes $P_{\mathcal{F}_0}$ of equiv sat. subsheaves $\mathcal{F}$ $\longleftrightarrow$ F generated by indices in $PP_{\mathcal{E}}$

if $\dim F = 2$:

$F = \mathrm{span}(e_3, e_1)$ $\qquad$ $P_{\mathcal{F}}$ is "bigger" than $P_{\mathcal{E}}$: $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$

and $\mathcal{E}$ isn't stable!

Thank you for
your attention!