On Toric Sarkisov Links from $\mathbb{P}^4$

- Motivation (MMP)
- Preliminaries
- Results :

28 686.

$/\mathbb{C}.$

$\longrightarrow$ Suppose ω is smooth uniruled projective variety

BCHM :  $\hookrightarrow$ $\boxed{\text{MMP}}$ $\longrightarrow$ Mori fibre space.

Ex: S = cubic Surface

$$S \downarrow \mathbb{P}^2$$

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-n)) \qquad n>1.$$

$\hookrightarrow$ Representative Not unique.

Goal: Study relations between end products of MMP

Defn: $\sigma : Y \to B$ surjective morphism of normal projective varieties $\sigma_* \mathcal{O}_Y = \mathcal{O}_B$ is a *Mori fibre space* if

- Y has $\mathbb{Q}$-factorial terminal singularities
- $-K_Y$ is σ-ample (Fibres are Fano)
- $\dim B < \dim Y$, $\rho(Y) - \rho(B) = 1$.

Ex:

$\sigma : \mathbb{P}^n \to pt$

$\mathbb{P}^n$

$X_d \subseteq \mathbb{P}^n$ smooth, $n+1-d > 0$.

Absolute Fano

$\mathbb{F}_n$

$\downarrow$ $\quad \mathbb{P}^1 \leftarrow$ Fano.

$\mathbb{P}^1$

$\longrightarrow$ relative Fano

Thm: (Corti 95' 3-dim, Hacon-McKernan 2014 n-dimensional)

Any birational map between Mfs can be decomposed as a *finite* sequence of *Sarkisov Links*

Sarkisov Links from $\mathbb{P}^n$:

$$\begin{tikzcd}
 & T_0 \arrow[r, dashed, "\tau_0"] \arrow[dl, "\varphi"'] & T_1 \arrow[r, dashed] & \cdots \arrow[r, dashed, "\tau_{n-1}"] & T_n \arrow[dr, "\varphi'"] & \\
\mathbb{P}^n & & & & & X
\end{tikzcd}$$

φ: divisorial extraction.

τ_i - small $\mathbb{Q}$-factorial modification

$\varphi' \to \begin{cases} \text{Fibration} \\ \text{divisorial Contraction} \end{cases}$

Rule

Each step is $\mathbb{Q}$-factorial and terminal
$\underbrace{\hspace{6em}}_{\text{Mori Category.}}$

Rmk: Controlling the singularities along the way will obstruct the existence of the Sarkisov link.

Q: When do you have a Sarkisov Link?

Thm (Abban-Kaloghiros, ...)

$\varphi: T_0 \longrightarrow \mathbb{P}^n$ divisorial extraction.

T is Toric $\xrightarrow{\text{automatic}}$ $\begin{cases} T_0 \text{ is a Mori Dream Space} \\ \underline{T_i} \text{ are terminal } \mathbb{Q}\text{-factorial} \\ -K_T \in \operatorname{int}(\operatorname{Mov}(T)) \end{cases}$ $\Longleftrightarrow$ $\exists$ Sarkisov link & it is unique..

Problem:

$$\begin{array}{c} T \\ \varphi \downarrow \\ \mathbb{P}^4 \end{array}$$

φ - (a,b,c,d) weighted blowup of a point. Find (a,b,c,d) s.t. φ initiates a Sarkisov link.

Defn: (Weighted Blowup):

$(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{>0}$

$$\mathbb{C}^* \curvearrowright \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$$

$$(\lambda, (u, x_1, \dots, x_n)) \longmapsto (\lambda^{-1} u, \lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n)$$

$$T = \frac{\mathbb{C}^{n+1} - V(x_1, \dots, x_n)}{\mathbb{C}^*}$$

$$\varphi : T \longrightarrow \mathbb{C}^n \qquad \longleftarrow \text{The weighted blowup of } \mathbb{C}^n \text{ at } \underline{0}.$$

$$(u, x_1, \dots, x_n) \longmapsto (u^{\alpha_1} x_1, \dots, u^{\alpha_n} x_n)$$

$$T : \begin{array}{c} \begin{array}{cccc} u & x_1 & & x_n \end{array} \\ \left(\begin{array}{c|ccc} -1 & \alpha_1 & \dots & \alpha_n \end{array} \right) \end{array} \longleftarrow \text{Action on } \mathbb{C}^{n+1}$$

Can do the same on $\mathbb{P}^n_{x_0 : \dots : x_n}$

$$T : \begin{array}{c} \begin{array}{cccccc} u & x_0 & x_1 & & x_n \end{array} \\ \left(\begin{array}{cc|ccc} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & \alpha_1 & \dots & \alpha_n \end{array} \right) \end{array}$$

$\mathbb{C}^*$ defining $\mathbb{P}^n$

$\mathbb{C}^*$ defining the bu.

$$T := \frac{\mathbb{C}^{n+2} \setminus \mathbb{V}(u, x_0) \cup \mathbb{V}(x_1, \dots, x_n)}{\mathbb{C}^* \times \mathbb{C}^*}$$

weighted bu of $\mathbb{P}^n$ at $p_{x_0} = (1 : 0 : \dots : 0)$

Singularities of T

Ex:

$$T : \begin{array}{c} \begin{array}{ccccc} u & x_0 & x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 3 \end{array} \right) \end{array}$$

$(x_0 x_3 \neq 0) \simeq \frac{1}{3}(1, 1, 2)$ singularity.

$(x_0 x_3 \neq 0) \simeq \operatorname{Spec} \mathbb{C}\left[u, x_0 \ldots, x_3, \frac{1}{x_0}, \frac{1}{x_3}\right]^{\mathbb{C}^* \times \mathbb{C}^*}$

$\vdots$

$\simeq \operatorname{Spec} \mathbb{C}[xz, yz, z^3, x^3, x^2y, xy^2, y^3] \leftarrow$

$\simeq \operatorname{Spec} \mathbb{C}[x, y, z]^{\mu_3}$

$$\begin{array}{ccc} \mu_3 \curvearrowright \mathbb{A}^3 & \longrightarrow & \mathbb{A}^3 \\ (\varepsilon, x, y, z) & \longrightarrow & (\varepsilon x, \varepsilon y, \varepsilon^2 z) \end{array} \leftarrow \frac{1}{3}(1,1,2).$$

How to get the other T_i ?

Cones in T:

$N^1(T) \to$ vector space of $\mathbb{Q}$-divisor$/\equiv$

$\dim N^1(T) = \operatorname{rank} \operatorname{Pic}(T) = 2$

$\operatorname{Pic} T = \mathbb{Z} H + \mathbb{Z} E \qquad \underline{H = \varphi^* H}$

$\operatorname{Nef}(T) \subseteq \operatorname{Mov}(T) \subseteq \operatorname{Eff}(T) \subseteq N^1(T).$

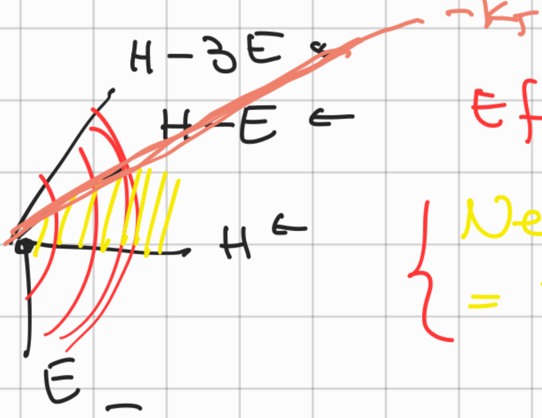
Ex:

$$\begin{array}{c} T \\ \Big\downarrow (1,1,3) \\ \mathbb{P}^3 \end{array}$$

$$T: \begin{array}{ccccc} u & x_0 & x_1 & x_2 & x_3 \end{array}$$
$$\left(\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 3 \end{array}\right)$$

$E : (u=0)$

$E \simeq \mathbb{P}(1,1,3)$



Eff(T)

Nef(T) = Mov(T)

$$-K_T = 4H - 4E = 4(H-E).$$

$$-K_T \notin \mathrm{int}(\mathrm{Mov}(T)).$$

Ex:

$$\begin{array}{c} T \\ \Big\downarrow (1,1,2,2) \\ \mathbb{P}^4 \end{array}$$

$$T: \begin{array}{cccccc} u & x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left(\begin{array}{cc|cccc} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & \underline{1} & \underline{1} & \underline{2} & \underline{2} \end{array}\right)$$

Mov(T).

Nef(T)

H−2E is movable. but <u>not</u> nef

$$\mathrm{Mov}(T) = \mathrm{Nef}(T) \cup \mathrm{Nef}(T')$$

Hu–Keel $\longrightarrow$ T' is the only SQM of T

$$\alpha = \alpha_{|H-E|} : T \longrightarrow \mathcal{F}$$

$$T: \begin{array}{c} \begin{array}{cccccc} u & x_0 & x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cc|cccc} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 2 & 2 \end{array}\right) \end{array}$$

$$(u, x_0, \dots, x_4) \longmapsto (\underbrace{x_1, x_2, u x_3, u x_4}_{\binom{1}{1}}, \underbrace{x_0 x_3, x_0 x_4}_{\binom{2}{2}})$$

$$S \subseteq T \xrightarrow{\alpha} \mathcal{F}$$

α contracts the locus $(x_3 = x_4 = 0)$

$$S \to \Gamma \simeq \mathbb{P}^1_{x_1 : x_2}$$

$$S = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right)$$

$\to \mathbb{P}^2$

$\to$ blowup.

$S = \mathbb{F}_1 =$ blowup of $\mathbb{P}^2$ at a point.

$$\begin{tikzcd} S \arrow[dr, "\alpha"'] & & S' \arrow[dl, "\beta"] \\ & \Gamma \subseteq \mathcal{F} & \end{tikzcd}$$

$S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$

Small $\mathbb{Q}$-factorial modification.

$$\begin{tikzcd} \mathbb{F}_1 \subseteq T \arrow[d, "{(1,1,2,2)}"'] \arrow[dr, "\alpha"] \arrow[rr, dashed, "\tau"] & & T' \supseteq \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "\beta"'] \arrow[d, "\varphi' ?"] \\ \mathbb{P}^4 & T & X \end{tikzcd}$$

$$\varphi' = \varphi'_{|H-2E|} : T' \longrightarrow \mathbb{P}^1$$

$$(u, x_0, \dots, x_4) \longmapsto (x_3, x_4)$$

Fibres $\simeq \mathbb{P}(1,1,1,2)$

φ' is a Mori fibre space.

$$T: \begin{array}{cccccc} u & x_0 & x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$\left(\begin{array}{cc|cccc} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 2 & 2 \end{array}\right)$$

$$\begin{tikzcd} F_1 \subseteq T \arrow[rr, dashed, "\tau"] \arrow[d, "{(1,1,2,2)}"'] \arrow[dr, "\alpha"] & & T' \supseteq \mathbb{P}^1\times\mathbb{P}^1 \arrow[dl, "\beta"'] \arrow[d, "F \cong \mathbb{P}(1,1,1,2)"] \\ \mathbb{P}^4 & T & \mathbb{P}^1 \end{tikzcd}$$

Thm: Let $\varphi: T \to \mathbb{P}^4$ be a toric (a,b,c,d)-weighted blowup of $\mathbb{P}^4$. Then φ initiates a Sarkisov Link iff (a,b,c,d) is one of $\underline{421}$ tuples up to permutation:

Thm: ($\mathbb{P}^3$)

$$\begin{tikzcd} T \arrow[d, "{(a,b,c)}"] \\ \mathbb{P}^3 \end{tikzcd}$$

4 tuples.

"Proof":

restricts the possibilities

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[d] & T' \arrow[d] \\ \mathbb{P}^4 \arrow[r, dashed] & \mathbb{P}^4(a,b,c,d,e) \end{tikzcd} \leftarrow \underline{28\,686} \rightarrow$$

< 421.

Kawakita

$E \quad T$

$$T \xrightarrow{(a,b,c)} \mathbb{P}^3$$

$(a,b,c) \longrightarrow (1,a,b)$ – weighted bus

s.t. $\gcd(a,b) = 1$.

$$E \simeq \mathbb{P}(1,a,b)$$

$(1,1,3)$