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Wall-crossing for Newton-Okounkov bodies

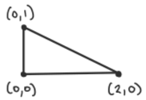
Wall Crossing for Newton-Okounkov bodies

joint work with Megumi Harada

Outline: ① Newton-Okounkov bodies
② Wall-Crossing for NO-bodies

The Newton polytope of $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$

is $\mathrm{Newt}(f) := \mathrm{conv}\{\alpha \mid c_\alpha \neq 0\}$.

$\mathrm{Newt}(3x_1^2 + x_2 - 1) =$ 

Bernstein-Khovanskii-Kuchnirenko Thm: $A \subseteq \mathbb{Z}^n$ finite

$L_A := \{ \sum_{\alpha \in A} c_\alpha x^\alpha \mid c_\alpha \in \mathbb{C} \}$. For a generic choice of

$f_1, \dots, f_n \in L_A$ the number of solutions to $f_1 = \cdots = f_n = 0$

in $(\mathbb{C}^*)^n$ is $n!\,\mathrm{vol}(\mathrm{conv}(A))$.

$\Delta \subseteq \mathbb{R}^n$ polytope $\rightsquigarrow X_\Delta \subseteq \mathbb{P}^s$ projective toric variety.

$$n!\,\mathrm{vol}(\Delta) = \deg(X_\Delta)$$

X, irreducible projective variety over $\mathbb{C}$

A, homogeneous coordinate ring of X

NO-bodies [Okounkov, Lazarsfeld-Mustaţă, Kaveh-Khovanskii]

valuation on $A \rightsquigarrow$ convex body Δ

$$\deg(X) = n!\,\mathrm{vol}(\Delta)$$

Thm [Anderson, 2013]: When Δ is a polytope of $\dim(\Delta) = \dim(X)$, there is a degeneration of X to X_Δ.

Thm [Harada-Kaveh, 2015]: When X is smooth there exists a full dimensional Hamiltonian torus action with moment map image Δ.

Cluster varieties

Equip $\mathbb{Z}^n$ w/ total order $\leq$.

A valuation is $\nu: A \setminus \{0\} \longrightarrow \mathbb{Z}^n$ st

① $\nu(f+g) \geq \max(\nu(f), \nu(g))$

② $\nu(fg) = \nu(f) + \nu(g)$

③ $\nu(c) = 0 \quad \forall c \in \mathbb{C}^*$

Example: $\nu(\sum c_\alpha x^\alpha) := \min\{\alpha \mid c_\alpha \neq 0\}$ is a valuation on $\mathbb{C}[x_1, \dots, x_n]$

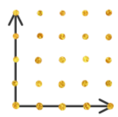


image of ν

Another example: X = hypersurface $y^2z - x^3 + 7xz^2 - 2z^3$

$M = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix}$

Valuation $\nu : \mathbb{C}[X] \longrightarrow \mathbb{Z}^2$

$$\sum c_{\alpha,\beta,\gamma} x^\alpha y^\beta z^\gamma \longmapsto \min\left\{ M \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \text{ st } c_{\alpha,\beta,\gamma} \neq 0 \right\}$$

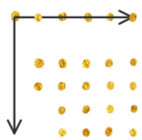
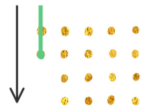


image of ν = semigroup generated by the columns of M

NO-body

$\Delta(X,\nu) = \overline{\mathrm{cone}(\mathrm{image}(\nu))} \cap \{x_1 = 1\}$



$\Delta(X,\nu)$ = convex hull of columns of M

$\Delta(X,\nu)$ is a polytope if $\mathrm{image}(\nu)$ is finitely generated.

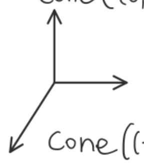
[Kaveh-Manon]: tropical geometry $\Leftrightarrow$ valuations w/ polytopal NO-bodies

I ideal, $\mathrm{trop}(I) := \left\{ w \in \mathbb{Q}^n \;\middle|\; \begin{array}{l} \mathrm{in}_w I \text{ contains} \\ \text{no monomials} \end{array} \right\}$

w/ fan structure having cones

$C_w := \overline{\{w' \in \mathrm{trop}(I) \mid \mathrm{in}_{w'} I = \mathrm{in}_w I\}}$

Example: $\mathrm{trop}(\langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle)$



$\mathrm{cone}((0,1,0), \pm\mathbb{1})$ $\quad \mathrm{in}_w I = \langle -x^3 + 7xz^2 - 2z^3 \rangle$

$\mathrm{cone}((1,0,0), \pm\mathbb{1})$ $\quad \mathrm{in}_w I = \langle y^2z - 2z^3 \rangle$

$\mathrm{cone}((-2,-3,0), \pm\mathbb{1})$ $\quad \mathrm{in}_w I = \langle y^2z - x^3 \rangle$

A cone C in $\mathrm{trop}(I)$ is prime if $\mathrm{in}_w I$ is prime for $w \in C^\circ$.

Thm [Kaveh-Manon]: Let C be a prime cone of $\mathrm{trop}(I)$.

① $\{u_1,\dots,u_r\} \subseteq C$ linearly ind, $r = \dim C$.

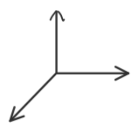
② M = mtx w/ rows $u_1,\dots,u_r$

Construct a valuation ν_C st

$\Delta(X,\nu_C)$ = convex hull of the columns of M

Example: Grassmannian $Gr(2,4)$

$I_{2,4} = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle$

$\mathrm{trop}(I_{2,4}) =$  $\times\ \mathcal{L} \subseteq \mathbb{R}^6$

$\mathcal{L}$: 4D vector space.

All cones are prime

C_1, C_2 prime cones of maximal dim.

$\rightsquigarrow \Delta(Gr(2,4),\nu_{C_1}),\ \Delta(Gr(2,4),\nu_{C_2})$.

Semigroups $\mathrm{image}(\nu_{C_1})$, $\mathrm{image}(\nu_{C_2})$

Maximal cones $\mathrm{trop}(I_{2,m})$ $\longleftrightarrow$ triangulations of labelled

m-gon

$\Phi(z_{12}, z_{13}, z_{14}, z_{23}, z_{34}) = (z_{12}, z_{14}, z_{23}, z_{24}, z_{34})$

$$z_{24} = \mathrm{trop}\left(\frac{z_{12}z_{34} + z_{14}z_{23}}{z_{13}}\right)$$

[Nohara-Ueda]

Geometric wall-crossing for NO-bodies

X projective variety

$A = \mathbb{C}[x_1,\dots,x_n]/I$

C_1, C_2 prime cones of $\mathrm{trop}(I)$ st

C_1, C_2 maximal dimension, and

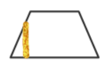
C_1, C_2 share a facet, C.

Then

$$\begin{tikzcd}[column sep=small]
\Delta(X,\nu_{C_1}) \arrow[dr, "\pi"] & & \Delta(X,\nu_{C_2}) \arrow[dl, "\pi"'] \\
& \Delta(X,\nu_C) &
\end{tikzcd}$$

where $\pi: \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$ projection

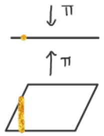
Thm [E.-Harada]: The

$\Delta(X,\nu_{C_1})$ 

fibers $\pi^{-1}(p) \cap \Delta(X, \nu_{c_1})$ and $\pi^{-1}(p) \cap \Delta(X, \nu_{c_2})$ are intervals of the same length.

$\Delta(X, \nu_c)$

$\Delta(X, \nu_{c_2})$



Thm [E.-Harada]: Obtain 2 piecewise linear maps Φ_S, Φ_F

$$\begin{tikzcd}
\Delta(X, \nu_{c_1}) \arrow[rr, "\Phi_{S \text{ or } F}"] \arrow[dr, "\pi"'] & & \Delta(X, \nu_{c_2}) \arrow[dl, "\pi"] \\
& \Delta(X, \nu_c) &
\end{tikzcd}$$

Remarks:

[Ilten-Manon]: geometric wall-crossing can be derived from the theory of complexity-1 T-varieties

[Ilten]: interpretation of geometric wall-crossing as a generalization of the combinatorial mutation of Akhtar-Coates-Galkin-Kasprzyk

[E.-Harada]: algebraic wall-crossing, i.e. bijection

$$\begin{tikzcd}
\text{image}(\nu_{c_1}) \arrow[rr] \arrow[dr] & & \text{image}(\nu_{c_2}) \arrow[dl] \\
& \text{image}(\nu_c) &
\end{tikzcd}$$

[E.-Harada, Bossinger-Mohammadi-Nájera Chávez]: For $Gr(2,m)$, Φ_F arises from cluster algebra structure.

Idea of proof:

① Since C_i is prime & maximal dim'l $\Rightarrow$ $in_{C_i} I$ is toric and cuts out the toric variety of $\Delta(X, \nu_{C_i})$.

Let $X_i = V(in_{C_i} I)$ and $Y = V(in_{C_1 \cap C_2} I)$.

② There exist flat families $\mathfrak{X}_1$, $\mathfrak{X}_2$ over $\mathbb{A}^1$ with generic fibers Y
special fibers X_1, X_2 (resp.)

③ There is a codimension-1 torus $T \curvearrowright Y$

T is a subtorus of $T_i \curvearrowright X_i$

The toric variety of a fiber $\pi^{-1}(p) \in \Delta(X, \nu_{C_i})$ is the GIT quotient of X_i by T at p.

④ T-action extends to $\mathcal{X}_1, \mathcal{X}_2$.

By taking the GIT quotient of $\mathcal{X}_i$ by T at p, we obtain 2 flat families

with generic fiber $Y /\!/_p T$

special fibers $X_i /\!/_p T$ (resp.)

⑤ Flat families preserve degree

$$\Rightarrow \deg(X_1 /\!/_p T) = \deg(Y /\!/_p T) = \deg(X_2 /\!/_p T)$$

$$\text{length of } \pi^{-1}(p) \cap \Delta(X, \nu_{C_1}), \quad \text{length of } \pi^{-1}(p) \cap \Delta(X, \nu_{C_2})$$