Automorphisms of Weighted Projective Hypersurfaces

/$\mathbb{C}$

$X_d \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$ a smooth hypersurface of dim. n and degree d

1) When is $\mathrm{Aut}(X)$ linear?

every automorphism comes from PGL_{n+2}

<u>Thm</u>: (Grothendieck-Lefschetz, Matsumura-Monsky, Chang)

Let $X_1 \cong X_2$ be an iso of hypersurfaces in $\mathbb{P}^{n+1}$, $n \geq 1$, Then it is linear unless

(1) $n=1$, $\{d_1, d_2\} = \{1,2\}$, (2) $n=1$, $d=3$,

(3) $n=2$, $d=4$

2) When is $\mathrm{Aut}(X)$ <u>finite</u>?

<u>Thm</u>: (Matsumura-Monsky, '64): If $n \geq 1$ and $d \geq 3$, then $\mathrm{Lin}(X)$ is finite.

automorphisms from PGL_{n+2}

Q: How do we <u>explicitly</u> bound $\mathrm{Lin}(X)$?

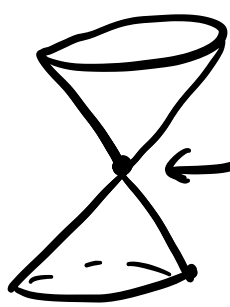
Goal: extend theorems to weighted projective space

$$\mathbb{P}(\underbrace{a_0, \dots, a_{n+1}}_{\text{weights}}) = (\mathbb{A}^{n+2} \setminus \{0\}) / \mathbb{C}^*$$

where $t \in \mathbb{C}^*$ acts by

$$t \cdot (x_0, \dots, x_{n+1}) = (t^{a_0} x_0, \dots, t^{a_{n+1}} x_{n+1})$$

<u>Ex</u>: $\mathbb{P}(\underbrace{1, 1, \dots, 1}_{n+2}) \cong \mathbb{P}^{n+1}$

$\mathbb{P}(2, 1, 1) =$ 

Assume that $\mathbb{P}$ is well-formed:

$$\gcd(a_0, \dots, \hat{a}_i, \dots, a_{n+1}) = 1$$

for each $i = 0, \dots, n+1$

Let $f = f(x_0, \ldots, x_{n+1})$ be homogeneous of weighted degree d ($\deg(x_i) = a_i$).

Then $X := \{f = 0\} \subseteq \mathbb{P}(a_0, \ldots, a_{n+1})$ is a hypersurface.

X is <u>quasismooth</u> if $\{f=0\} \subseteq \mathbb{A}^{n+2} \setminus \{0\}$ is smooth.

$$\mathbb{P}(a_0, \ldots, a_{n+1}) = \operatorname{Proj} S$$

$$S = \mathbb{C}[\underset{\text{wt } a_0}{\underset{\uparrow}{x_0}}, \ldots, \underset{\text{wt } a_{n+1}}{\underset{\uparrow}{x_{n+1}}}]$$

<u>graded</u> automorphisms

<u>Prop</u>: $\operatorname{Aut}(\mathbb{P}(a_0, \ldots, a_{n+1})) = \operatorname{Aut}(S)/H$

"scalar transformations"

$$(x_0, \ldots, x_{n+1}) \longmapsto (t^{a_0} x_0, \ldots, t^{a_{n+1}} x_{n+1})$$

<u>Ex</u>: $\operatorname{Aut}(\mathbb{P}^{n+1}) = PGL_{n+2}$

- Call elements of $\operatorname{Aut}(S)$ "linear"

<u>Ex</u>: $\mathbb{P}(\underset{x}{4}, \underset{y}{3}, \underset{z}{1})$

$$x \longmapsto x + yz + z^4$$
$$y \longmapsto -y + z^3$$
$$z \longmapsto 2z$$

(§1) Linearity

- saw that $\mathrm{Aut}(X) = \mathrm{Lin}(X)$
  for most $X_d \subseteq \mathbb{P}^{n+1}$ smooth

<u>Theorem A</u>: (E., 2023)
Let $X_d \subseteq \mathbb{P}(a_0, \ldots, a_{n+1})$, $X'_{d'} \subseteq \mathbb{P}(a'_0, \ldots, a'_{n+1})$
be two well-formed, quasismooth hypersurfaces such that $d \neq a_i$ for any i and either

(1) $n \geq 3$, or (2) $n = 2$, $a_0 + a_1 + a_2 + a_3 \neq d$

If $g : X' \xrightarrow{\cong} X$ is an iso, then
$d = d'$, $\{a_0, \ldots, a_{n+1}\} = \{a'_0, \ldots, a'_{n+1}\}$ and g is linear.

Idea: $\mathrm{Cl}(X) \cong \mathbb{Z}$, $\mathrm{Cl}(X') \cong \mathbb{Z}$
NTS $\mathrm{Cl}(X) \cong \mathrm{Pic}^{\mathbb{G}_m}(\text{affine cone over } X)$

<u>Remark</u>: Przyjalkowski–Shramov & others have had partial results for weighted comp. int.'s

<u>Ex</u>: (failure of uniqueness of embeddings for $n=1$)

$$R(X,D) := \bigoplus_{i=0}^{\infty} H^0(X, iD)$$

a) Let X be sm., genus 1 curve
$p \in X$ a rational point

$$R(X,p) \cong k[\underset{3}{x_1}, \underset{2}{x_2}, \underset{1}{x_3}]/(f_6)$$

$$\Rightarrow \quad X = X_6 \subseteq \mathbb{P}(3,2,1)$$

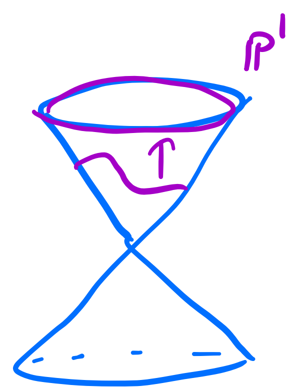
Weierstrass rep of elliptic curve

$$x_1^2 = x_2^3 + a x_2 x_3^4 + b x_3^6.$$

$$R(X, 2P) \cong k[\underset{2}{y_1}, \underset{1}{y_2}, \underset{1}{y_3}]/(g_4)$$

$$\Rightarrow \quad X = X_4 \subseteq \mathbb{P}(2,1,1)$$

double cover of $\mathbb{P}^1$

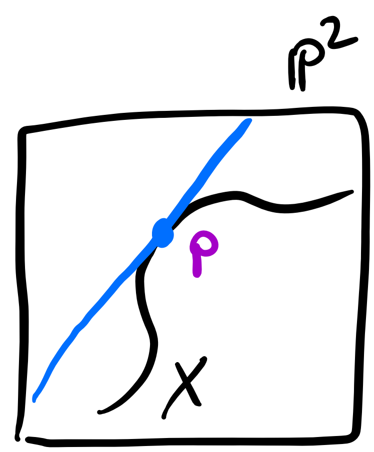


$$R(X, 3P) \cong k(\underset{1}{z_1}, \underset{1}{z_2}, \underset{1}{z_3})/(h_3)$$

cubic plane curve

b) $X_4 \subseteq \mathbb{P}(1,1,1)$ smooth

X is tangent to a line
with order 4 at p



$R(X,\rho) \Rightarrow$

$$X_{12} \subseteq \mathbb{P}(4,3,1)$$

Idea for Thm A:

Show that $g : X' \longrightarrow X$

maps $\mathcal{O}_{X'}(1)$ to $\mathcal{O}_X(1)$

(Grothendieck-Lefschetz)
$n \geq 3$

(§2) Finiteness

<u>Theorem B</u> (E., 2023)

Let $X_d \subseteq \mathbb{P}(a_0, \dots, a_{n+1})$ be well-formed, quasismooth. $\mathrm{Lin}(X)$ is finite iff:

(1) $d > 2\max\{a_0, \dots, a_{n+1}\}$, or

(2) $d = 2\max\{a_0, \dots, a_{n+1}\}$ but only

$$a_0 = \frac{d}{2}$$

If neither (1) nor (2) holds,
$\mathrm{Lin}(X)$ is infinite and X is rational.

Idea: if X is a quadric in some variables $\Rightarrow$ $\mathrm{Lin}(X)$ infinite

- Proof: computing $\dim(\mathrm{Lie}(\mathrm{Lin}(X))) = 0$ if (1) or (2) holds

Q: How do we bound $\mathrm{Lin}(X)$ explicitly?

- Bott, Tate (1961): proved $\exists\, k_{n,d}$

$$|\mathrm{Lin}(X)| \leq k_{n,d}$$

$X_d \subseteq \mathbb{P}^{n+1}$ smooth

- Howard, Sommese (1981): proved $\exists\, k_n$ such that $|\mathrm{Lin}(X)| \leq k_n d^{n+1}$, $d \geq 3$

$X_d \subseteq \mathbb{P}^{n+1}$ smooth

- k_n not explicit

<u>Theorem C</u>: (E., 2023)

For each $n \geq 1$, there exists a constant C_n such that: for any well-formed, quasismooth $X_d \subseteq \mathbb{P}(a_0, \dots, a_{n+1})$ of dim. n, if $\mathrm{Lin}(X)$ is finite, then

$$|\mathrm{Lin}(X)| \leq C_n \frac{d^{n+1}}{a_0 \cdots a_{n+1}}.$$

can compute an explicit value: $\sim(2n)!$ suffices

Expectation: $C_n = (n+2)!$ usually works (works for n large enough)

Prop: $C_1 = \frac{21}{2}$ is optimal

<u>Proof idea</u>:

Step 1: Translate to a statement about graded rings

If $H \leq \mathrm{Aut}(S)$ is defined as

$$H = \{ h : h \cdot f = f \},$$

defines hypersurface X

then Theorem $\Longleftrightarrow$ $|H| \leq C_n \frac{d^{n+2}}{a_0 \cdots a_{n+1}}$

Step 2: Reduce to abelian groups

Thm: (Jordan, 1878)

There exists a constant J_N ← Jordan const. for GL_N

such that for any finite group

$H \leq GL_N(\mathbb{C})$, there exists a normal abelian subgroup $A \leq H$ such that $[H:A] \leq J_N$.

Thm: (Collins, 2007)

When $N \geq 71$, $J_N = (N+1)!$

J_N achieved by standard rep. of S_{N+1} in $GL_N(\mathbb{C})$

Lemma: Let $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$ be a weighted graded poly. ring. Then the Jordan constant of $\mathrm{Aut}(S)$ is uniformly bounded by C_n, indep. of weights.

If A is abelian, get bound $|A| \leq \frac{d^{n+2}}{a_0 \cdots a_{n+1}}$.

<u>Ex</u>: $\mathbb{P}(a,b,c)$

a,b,c are distinct

Then every finite subgroup of $\mathrm{Aut}(\mathbb{P})$ is conjugate to an abelian group.