Kanev and Todorov type surfaces in toric 3-folds

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Outline

Motivation

Given a finite subset $A \subset \mathbb{Z}^n$ and a Laurent polynomial

$$f = \sum_{m \in A} a_m x^m, \quad a_m \in \mathbb{C}$$

the convex hull Δ of A is called the Newton polytope of f.

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the convex hull Δ of A is called the Newton polytope of f.

Example: Let $f(x_1, x_2) := a_1 + a_2 x_1^3 + a_3 x_2^3 + a_4 x_1 x_2$. Then

$$\Delta = \langle (0,0), (3,0), (0,3) \rangle.$$

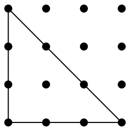

Figure: The Newton polytope Δ of a plane cubic

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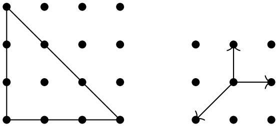

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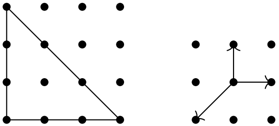

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We denote by $\mathbb{P}_\Delta$ the n-dim. (projective) toric variety to the normal fan of Δ and for a fan Σ by $\mathbb{P}_\Sigma$ the toric variety to the fan Σ.

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More generally for an n-dimensional fan Σ we write Z_Σ or $Z_{\Sigma,f}$ for the closure of Z_f in $\mathbb{P}_\Sigma$ (this notation will become obvious).

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We call (and always assume) f nondegenerate with respect to Δ, if Z_f is smooth and Z_Δ intersect the toric strata of $\mathbb{P}_\Delta$ transversally.

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To realize just the first point would be easy, since we could choose a toric resolution of singularities of $\mathbb{P}_\Delta$. But in fact there is a more intrinsic method to realize both points at the same time.

The Fine interior

We start with a lattice polytope

$$\Delta = \{x \in M_{\mathbb{R}} | \, \langle x, \nu_i \rangle \geq r_i\}, \quad \nu_i \in N, \; r_i \in \mathbb{Z}.$$

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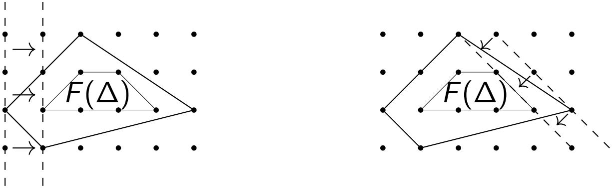
Define the Fine interior

$$F(\Delta) := \{x \in M_{\mathbb{R}} | \langle x, \nu \rangle \geq \mathrm{ord}_{\Delta}(\nu) + 1,\ \nu \in N \setminus \{0\}\}$$

The Fine interior

Concretely: Take any hyperplane touching Δ and move it one step into the interior of Δ. Then they cut out the Fine interior.



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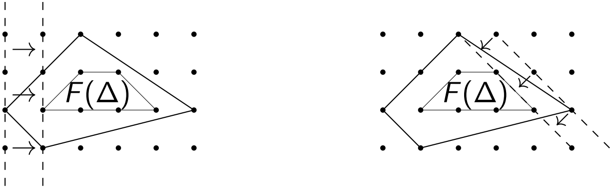

Figure: Illustration of the construction of the Fine interior $F(\Delta)$ from Δ.

$F(\Delta)$ always contains the convex span of the interior lattice points of Δ with equality in dimension 2. In dimension at least three $F(\Delta)$ is in general just a rational polytope.

Examples

For Δ reflexive, i.e.

$$\Delta = \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geq -1\}, \quad \nu_i \in N$$

we have $F(\Delta) = \{0\}$.

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There are 674 688 three-dim. canonical Fano polytopes, and dim $F(\Delta)$ happens to be 0, 1 or 3 for them. There are just 49 such polytopes with dim $F(\Delta) = 3$.

Canonical Fano 3-topes Δ with dim $F(\Delta) = 3$

Known result: ([Sch18]): All facets of Δ have distance 1 to 0 except from one facet Δ_{can}, which has distance 2. (For $H := \{x \in M_{\mathbb{R}} | \langle x, \nu \rangle = r\}$, the integer $|r|$ is called the lattice distance of H to 0).

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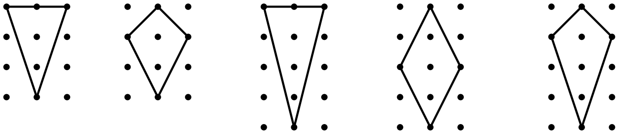

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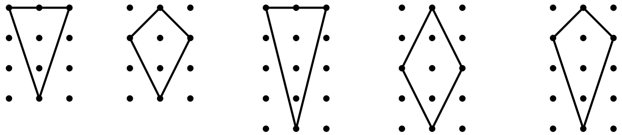

Figure: The 5 different types for Δ_{can}

Observation: In every of the 5 classes there is a unique maximal polytope among the 49 polytopes, with respect to inclusion of sets.

The support $S_F(\Delta)$

Let Δ be a lattice polytope. We define the support $S_F(\Delta)$ of the Fine interior:

$$S_F(\Delta) := \{\nu \in N \setminus \{0\} | \operatorname{ord}_{F(\Delta)}(\nu) = \operatorname{ord}_{\Delta}(\nu) + 1\}$$

that is $S_F(\Delta)$ consists of the normal vectors $\nu \in N$ to those hyperplanes H_ν such that H_ν touches Δ and H_ν touches $F(\Delta)$ after replacing it by one step into the interior direction.

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Theorem ([Bat20]): Let $\Sigma_\Delta[1] = \{\nu_1, ..., \nu_k\}$, then

$$S_F(\Delta) \subset \langle \nu_1, ..., \nu_k \rangle.$$

In particular we get for the cardinality $|S_F(\Delta)| < \infty$.

Toric varieties with terminal sing.

There is the following criterion due to M. Reid :
The toric variety $\mathbb{P}_\Sigma$ to a complete simplicial fan Σ has at most terminal singularities iff for every maximal-dim. cone $\sigma \in \Sigma$ with say $\sigma[1] = \{\nu_1, \ldots, \nu_n\}$ we have

$$\langle 0, \nu_1, \ldots, \nu_n \rangle \cap N = \{0, \nu_1, \ldots, \nu_n\}$$

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We show for $n = 3$ that if $F(\Delta) \neq \emptyset$ and Σ is simplicial with $\Sigma[1] = S_F(\Delta)$, then

- $\mathbb{P}_\Sigma$ has terminal sing.
- $K_{\mathbb{P}_\Sigma} + Z_\Sigma$ is nef.

The hypersurface Z_Σ then will also have at most terminal singularities and K_{Z_Σ} nef.

Toric varieties with terminal sing.

The first point could be seen combinatorially: Given $\nu_1, \nu_2, \nu_3 \in S_F(\Delta)$ spanning a cone of Σ. Assume

$$H_{\nu_i, b_i} := \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle = b_i\}$$

touches Δ and thus H_{ν_i, b_i+1} touches $F(\Delta)$.

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$$H_{\nu_1,b_1} \cap H_{\nu_2,b_2} \cap H_{\nu_3,b_3} \cap \Delta = \{pt\} \neq \emptyset.$$

Further (since $F(\Delta) \neq \emptyset$) we claim that

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For else there would be another hyperplane $H_{\mu,a}$ touching Δ, such that $H_{\mu,a+1}$ touches $F(\Delta)$ and $H_{\mu,a+1}$ prevents q to lie in $F(\Delta)$. But then clearly $\mu \in S_F(\Delta)$ and due to convexity of $F(\Delta)$

$$\mu \in \text{Cone}(\nu_1, \nu_2, \nu_3) \cap S_F(\Delta) = \{\nu_1, \nu_2, \nu_3\}.$$

Toric varieties with terminal sing.

Let now

$$N \ni \nu := \sum_{i=1}^{3} a_i \nu_i \in \langle 0, \nu_1, \nu_2, \nu_3 \rangle.$$

This could be easily restricted to $\nu \in \langle \nu_1, \nu_2, \nu_3 \rangle$. We show $\nu \in S_F(\Delta)$: Choose

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$$H_{\nu, \sum a_i b_i} := \{x \in \Delta | \langle x, \nu \rangle = a_1 \langle x, \nu_1 \rangle + a_2 \langle x, \nu_2 \rangle + a_3 \langle x, \nu_3 \rangle = \sum_{i=1}^{3} a_i b_i \}$$

Then $H_{\nu, \sum a_i b_i}$ touches Δ and $H_{\nu, \sum a_i b_i + 1}$ touches $F(\Delta)$: It obviously contains $F(\Delta)$ and $q \in H_{\nu, \sum a_i b_i + 1} \cap F(\Delta)$.

The divisor $Z_\Sigma + K_{\mathbb{P}_\Sigma}$ is nef

The second point: With D_i the toric divisor to the ray ν_i we have

$$Z_\Sigma \sim_{lin} - \sum_{\nu_i \in \Sigma[1]} \mathrm{ord}_\Delta(\nu_i) D_i, \quad K_{\mathbb{P}_\Sigma} = - \sum_{\nu_i \in \Sigma[1]} D_i$$

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But since $\Sigma[1] = S_F(\Delta)$ we get

$$Z_\Sigma + K_{\mathbb{P}_\Sigma} \sim_{lin} - \sum_{\nu_i \in S_F(\Delta)} (\mathrm{ord}_\Delta(\nu_i) + 1) D_i = - \sum_{\nu_i \in S_F(\Delta)} \mathrm{ord}_{F(\Delta)}(\nu_i) D_i$$

In other words to the divisor $Z_\Sigma + K_{\mathbb{P}_\Sigma}$ is associated the polytope $F(\Delta)$. By this it follows easily that $Z_\Sigma + K_{\mathbb{P}_\Sigma}$ is a ($\mathbb{Q}$-Cartier) nef divisor.

Motivation canonical closure

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Since refinements of fans induces birational toric morphisms such a property would be desirable. For this we have to introduce the canonical closure

The canonical closure

The canonical closure $C(\Delta)$ is defined by

$$C(\Delta) := \{x \in M_{\mathbb{R}} | \langle x, \nu \rangle \geq \mathrm{ord}_{\Delta}(\nu) \quad \forall \nu \in S_F(\Delta)\}$$

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Remark: We have $F(\Delta) \subset \Delta \subset C(\Delta)$, $F(C(\Delta)) = F(\Delta)$ and

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Under the 49 polytopes there are 29 canonically closed polytopes. In particular the maximal polytopes among these polytopes are canonically closed.

Examples: First class

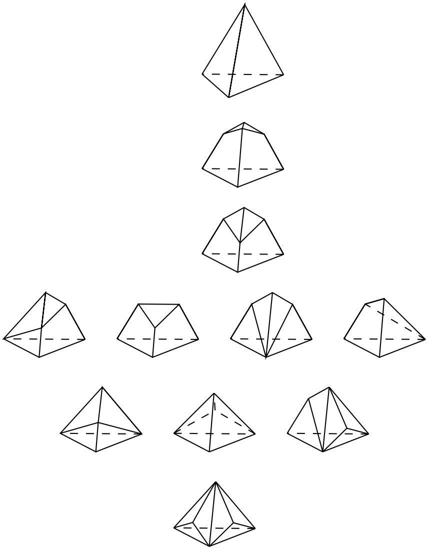

Figure: The 11 canonically closed polytopes out of 20 polytopes in the first class.

The Minkowski sum

Consider also the Minkowski sum

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Result ([Bat20, Thm.4.3]): We still have $\Sigma_{\tilde{\Delta}}[1] \subset S_F(\Delta)$.
In our cases, where $F(\Delta)$ is full-dimensional, this is elementary, since then obviously $\Sigma_{F(\Delta)}[1] \subset S_F(\Delta)$.
The fan $\Sigma_{\tilde{\Delta}}$ will already be good enough such that $\mathbb{P}_{\tilde{\Delta}}$ and with it $Z_{\tilde{\Delta}}$ have canonical singularities .

Applications to toric varieties

We get birational toric morphisms (we always assume $F(\Delta) \neq \emptyset$)

$$\begin{array}{ccccc} & & \mathbb{P}_{\Sigma} & & \\ & & \downarrow \pi & & \\ & & \mathbb{P}_{\tilde{\Delta}} & & \\ & \rho \swarrow & & \searrow \theta & \\ \mathbb{P}_{C(\Delta)} & & & & \mathbb{P}_{F(\Delta)} \end{array}$$

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Observation for our examples: For our polytopes we have

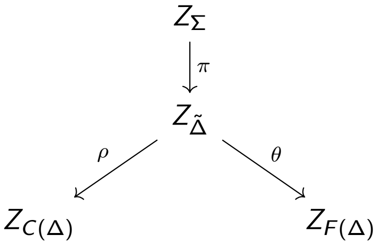
$$\Sigma_{\tilde{\Delta}}[1] = \Sigma_{C(\Delta)}[1]$$

This means that ρ is an isomorphism in codimension one. For Δ maximal we additionally have

$$\Sigma_{\tilde{\Delta}} = \Sigma_{F(\Delta)}.$$

Construction of minimal/canonical models

We take the closures Z_Σ, $Z_{\tilde{\Delta}}$, $Z_{C(\Delta)}$ and $Z_{F(\Delta)}$ of Z_f and get a diagram of induced birational morphisms



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Z_{C(\Delta)} & & & & Z_{F(\Delta)}
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For our polytopes $\rho : Z_{\tilde{\Delta}} \to Z_{C(\Delta)}$ is an isomorphism and $Z_{F(\Delta)}$ gets a canonical model of a surface of general type ([Gie21]).

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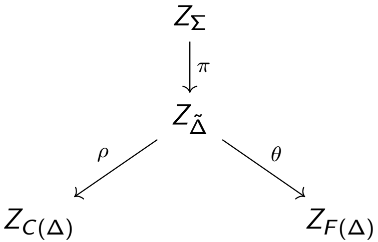

For our polytopes $\rho : Z_{\tilde{\Delta}} \to Z_{C(\Delta)}$ is an isomorphism and $Z_{F(\Delta)}$ gets a canonical model of a surface of general type ([Gie21]).
Result ([Bat20]): In arbitrary dimensions Z_Σ has at most terminal sing. with K_{Z_Σ} nef, i.e. Z_Σ gets a minimal model, and $Z_{\tilde{\Delta}}$ has at most canonical sing, $\pi : Z_\Sigma \to Z_{\tilde{\Delta}}$ is crepant.

The Kodaira dimension

Result ([Bat20]): For the Kodaira-dimension $\kappa(Z_{\tilde{\Delta}})$ of $Z_{\tilde{\Delta}}$ we have:

$$\kappa(Z_{\tilde{\Delta}}) = \min(\dim\ F(\Delta), n-1).$$

Thus for $n = 3$

$$\kappa(Z_{\tilde{\Delta}}) = \min(\dim\ F(\Delta), 2),$$

and our examples of surfaces are of maximal Kodaira dimension 2.

Adjunction

Result ([Bat20]): Quite generally in dimension n, $\theta \circ \pi : Z_\Sigma \to Z_{F(\Delta)}$ is given by $|m(K_{Z_\Sigma} + Z_\Sigma)|$ for $m \gg 0$ and thus by the adjunction formula

$$(Z_\Sigma + K_{\mathbb{P}_\Sigma})_{|Z_\Sigma} = K_{Z_\Sigma}$$

induces the Iitaka fibration for toric hypersurfaces. In fact m could be chosen as

$$m := \text{ind } F(\Delta) := \min\{n \in \mathbb{N}_{\geq 1} | \, n \cdot F(\Delta) \text{ is a lattice polytope}\}.$$

The refinements Σ of $\Sigma_{\tilde{\Delta}}$ and $\Sigma_{\tilde{\Delta}}$ of $\Sigma_{F(\Delta)}$

Observation: In all examples the refinements between $\Sigma_{F(\Delta)}$ and Σ happen on only one 3-dimensional cone σ of $\Sigma_{F(\Delta)}$.

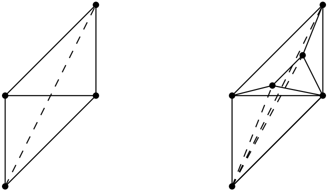

Figure: On the left is pictured σ in the first class and on the right the refinement of σ in $\Sigma_{\tilde{\Delta}}$.

This allows us to draw pictures of a cross section of this cone.

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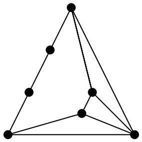

Figure: Cross section of σ for the above cone σ

The subdivision of the cross section of σ shows the fan $\Sigma_{\tilde{\Delta}}$.
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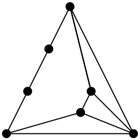

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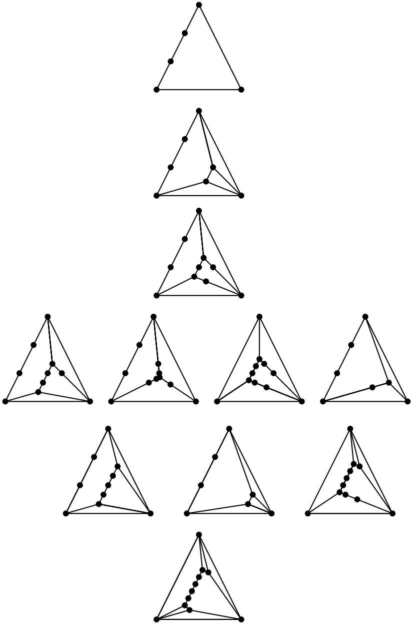
There might be more rays in $S_F(\Delta)$ but they lie within a 3-dimensional cone of $\Sigma_{\tilde{\Delta}}$ and are irrelevant for the minimal model Z_Σ.

Pictures of a cross section of σ



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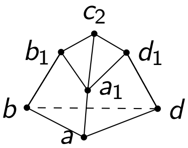

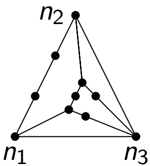

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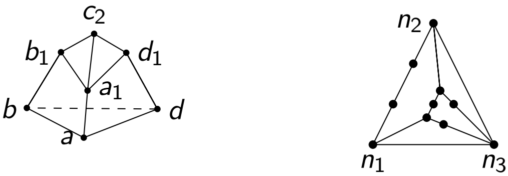

Figure: One polytope on the left and the cross section of σ on the right.

The cone spanned by n_1 and n_2 corresponds to the edge $\langle a, a_1 \rangle$.
$a - a_1 = (2, 1, -1)$ is primitive $\Rightarrow Z_{\tilde{\Delta}}$ intersects the toric stratum in one point.
$n_1 - n_2 = 3 \cdot (\text{prim. lattice vector}) \Rightarrow$ we get one singularity of type A_2 on $Z_{\tilde{\Delta}}$ from the edge $\langle a, a_1 \rangle$.

Interpretation of the pictures (3)

Theorem: The closure of Z_f in $\mathbb{P}_{F(\Delta)}$ has an ADE-singularity at the torus fixed point to σ. The points in the interior of σ build the vertices of the Dynkin diagram to this singularity.

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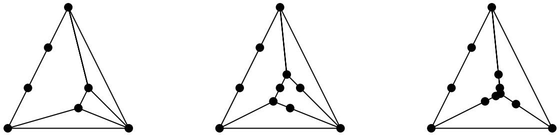

Figure: In the left picture we get an A_2 singularity at the torus fixed point to σ, in the middle picture an A_5 singularity and in the right an E_6 singularity.

The plurigenera

Result (Giesler,unpublished): Let $\Delta \subset M_{\mathbb{R}}$ be an n-dim. lattice polytope with $\dim F(\Delta) = k \geq 0$. Then for $X := Z_{\Sigma}$ or $X := Z_{\tilde{\Delta}}$ the plurigenera $P_m(X) := h^0(X, mK_X)$ of X are given by $(m \geq 2)$

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$$P_m(X) = \begin{cases} l(m \cdot F(\Delta)) - l^*((m-1) \cdot F(\Delta)), & k = n \\ l(m \cdot F(\Delta)) + l^*((m-1) \cdot F(\Delta)), & k = n-1 \\ l(m \cdot F(\Delta)) & k < n-1, \end{cases} .$$

where for a polytope $P \subset M_{\mathbb{R}}$: $l(P) := |P \cap M|$ and $l^*(P)$ denotes the number of interior lattice points of P.

Plurigenera

Proof: Without restriction let $X := Z_\Sigma$. Since

$$H^1(\mathbb{P}_\Sigma, m(K_{\mathbb{P}_\Sigma} + X)) = 0$$

for the nef divisor $m(K_{\mathbb{P}_\Sigma} + X)$ we get an ideal sheaf sequence

$$0 \to H^0(\mathbb{P}_\Sigma, (m-1)(K_{\mathbb{P}_\Sigma} + X) + K_{\mathbb{P}_\Sigma}) \to H^0(\mathbb{P}_\Sigma, m(K_{\mathbb{P}_\Sigma} + X))$$
$$\to H^0(X, mK_X) \to H^1(\mathbb{P}_\Sigma, (m-1)(K_{\mathbb{P}_\Sigma} + X) + K_{\mathbb{P}_\Sigma}) \to 0$$

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To $m(K_{\mathbb{P}_\Sigma} + X)$ is associated the polytope $mF(\Delta)$, which counts the global sections, i.e.

$$h^0(\mathbb{P}_\Sigma, m(K_{\mathbb{P}_\Sigma} + X)) = |m \cdot F(\Delta) \cap M|$$

Plurigenera

Continue the proof: By Serre duality for the $\mathbb{Q}$-Cartier divisor $(m-1)(K_{\mathbb{P}_\Sigma}+X)+K_{\mathbb{P}_\Sigma}$ and a vanishing result ([CLS11, Thm.9.2.7]) we get

$$H^0(\mathbb{P}_\Sigma,(m-1)(K_{\mathbb{P}_\Sigma}+X)+K_{\mathbb{P}_\Sigma}) \cong H^n(\mathbb{P}_\Sigma,(1-m)(K_{\mathbb{P}_\Sigma}+X))^*$$
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$$=\begin{cases} l^*((m-1)F(\Delta)), & \dim\ F(\Delta)=n-1 \\ 0, & \dim\ F(\Delta)\neq n-1 \end{cases}$$

The result follows.

The geometric genus and the irregularity

The geometric genus of Z_Σ is given by

$$p_g(Z_\Sigma) := h^0(Z_\Sigma, K_{Z_\Sigma}) = l^*(\Delta)$$

where $l^*(\Delta)$ denotes the number of interior lattice points of Δ.

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Example: In our examples we get

$$p_g(Z_\Sigma) = 1, \quad q(Z_\Sigma) = 0$$

since Δ is 3-dimensional and canonical ($l^*(\Delta) = 1$).

The canonical divisor

The geometric genus $p_g(K_{Z_\Sigma})$ could be read off from the facet Δ_{can} of Δ with distance 2 to the origin as $p_g(K_{Z_\Sigma}) = l^*(\Delta_{can})$.

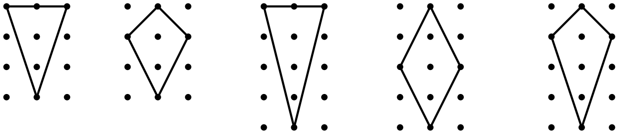

Figure: The 5 different types for Δ_{can}

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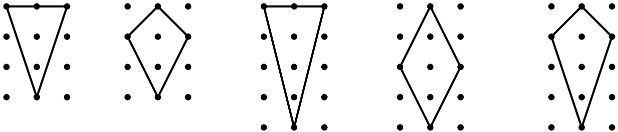

Figure: The 5 different types for Δ_{can}

We get minimal surfaces Z_Σ with

$$p_g(Z_\Sigma) = 1, \quad q(Z_\Sigma) = 0, \quad p_g(K_{Z_\Sigma}) \in \{2, 3\}$$

The condition $p_g(K_{Z_\Sigma}) \in \{2, 3\}$ is equivalent to $K_{Z_\Sigma}^2 \in \{1, 2\}$ by the adjunction formula.

Kanev and Todorov type surfaces

Surfaces with

$$p_g(Z_\Sigma) = 1, \quad K^2_{Z_\Sigma} = 1$$

are called Kanev surfaces and surfaces with

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$$P_m(Z_\Sigma) = 2 + \frac{m(m-1)}{2} K^2_{Z_\Sigma}.$$

Thus we obtain e.g. for Kanev surfaces ($K^2_{Z_\Sigma} = 1$) the identity ($m \geq 2$):

$$l(m \cdot F(\Delta)) - l^*((m-1)F(\Delta)) = 2 + \frac{m(m-1)}{2}.$$

The number of moduli

From a Newton polytope Δ we derive a family of hypersurfaces by varying the coefficients $(a_m)_{m \in M \cap \Delta}$ such that

$$f = \sum_{m \in \Delta \cap M} a_m x^m$$

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We are asking for the number of moduli of such a family. Let us make this precise

The number of moduli

Let $L(\Delta)$ be the convex span of the lattice points $M \cap \Delta$ and $U_{reg}(\Delta) \subset L(\Delta)$ be the set of nondegenerate Laurent polynomials (with Newton polytope Δ). We obtain a family of minimal models

$$\mathcal{X}_\Sigma := \{(y, f) \in \mathbb{P}_\Sigma \times \mathbb{P}U_{reg}(\Delta) |\, y \in Z_{\Sigma,f}\}$$

with natural projection $pr_2 : \mathcal{X}_\Sigma \to \mathbb{P}U_{reg}(\Delta)$.

The number of moduli

If $H^0(Z_\Sigma, T_{Z_\Sigma}) = H^2(Z_\Sigma, T_{Z_\Sigma}) = 0$, then by deformation theory there exists a universal deformation $\mathcal{X} \to S$ with S smooth and with ($X_f := Z_{\Sigma,f}$ the fibre over f)

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$$T_{S,f} \cong H^1(X_f, T_{X_f})$$

There is a suitable homomorphism $\kappa : T_{U_{reg}(\Delta),f} \to H^1(X_f, T_{X_f})$, the so called Kodaira-Spencer map , which connects these two families and we define:

Number of moduli $=$ dim $Im(\kappa)$.

The number of moduli

Result (Giesler to appear): For our examples of hypersurfaces in toric 3-folds we have

$$\dim \ker(\kappa) = \dim \mathrm{Aut}(\mathbb{P}_\Sigma) = \dim \mathrm{Aut}(\mathbb{P}_{\tilde{\Delta}})$$

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Thus

$$\dim Im(\kappa) = |M \cap \Delta| - 1 - \dim \operatorname{Aut}(\mathbb{P}_{\tilde{\Delta}}).$$

The number $\dim \operatorname{Aut}(\mathbb{P}_{\tilde{\Delta}})$ could be determined from the rays of the normal fan of $\tilde{\Delta}$.

The period map and its derivative

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We may define a period map for our families:

$$\begin{aligned} \mathcal{P} : U_{reg} &\to \text{ Period domain} \\ f &\mapsto H^{2,0}(Z_{\Sigma,f}) \end{aligned}$$

We will not specify the period domain in more detail here.

The period map and its derivative

Consider the derivative $d\mathcal{P}$ of $\mathcal{P}$: By results of Griffiths this derivative factors as follows ($X_f := Z_{\Sigma,f}$)

$$T_f U_{reg}(\Delta) \xrightarrow{\kappa} H^1(X_f, T_{X_f}) \xrightarrow{\Phi} \mathrm{Hom}(H^0(X_f, \Omega^2_{X_f}), H^1(X_f, \Omega^1_{X_f}))$$

with κ the Kodaira-Spencer map and Φ the homomorphism induced by contraction and cup-product.

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The infinitesimal Torelli theorem asks if $\Phi_{|Im\,\kappa}$ is injective.

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Result: For our examples of Kanev surfaces dim $\ker(\Phi) = 2$ and for Todorov type surfaces dim $\ker(\Phi) = 3$. In particular these surfaces fail the infinitesimal Torelli theorem.

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We already saw that the dimension dim $ker(\kappa)$ could be determined from the normal fan of $\tilde{\Delta}$.
The we use the formula

$$dim\, ker(d\mathcal{P}) = dim\, ker(\kappa) + dim\, ker(\Phi_{|Im\,\kappa})$$

to determine dim $\ker(\Phi_{|Im\,\kappa})$.

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