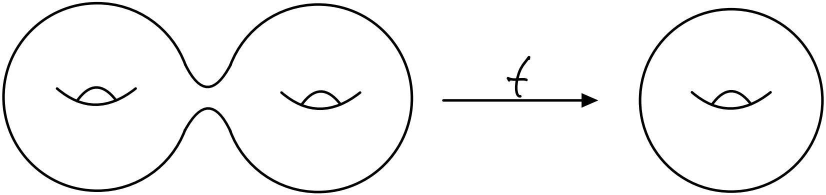
The tropical geometry of monotone Hurwitz numbers

Marvin Anas Hahn
(w/J.W. van Ittersum, F. Leid, R. Kramer and D. Lewanski)

Nottingham University
Online Algebraic Geometry Seminar, 20.01.22

Introduction and context

Hurwitz numbers: Important enumerative invariants in algebraic geometry



Introduction and context

Hurwitz numbers: Important enumerative invariants in algebraic geometry

Introduced by Hurwitz in the 1890s.

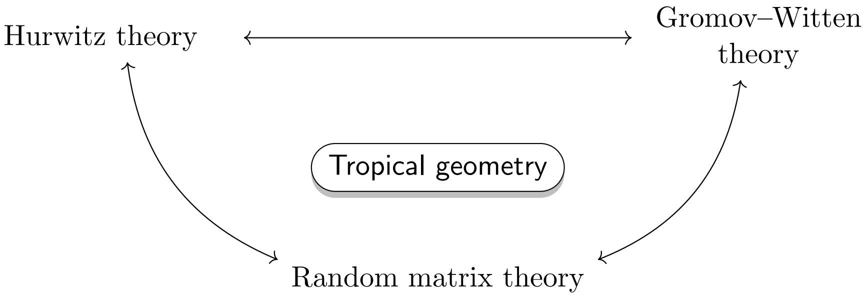
Rapid developments since 1990s due to deep connections with **Gromov–Witten theory** and **string theory**.

Connections to other fields

Introduction and context

Today: Focus on *monotone Hurwitz numbers* from random matrix theory.

Progress report: programme towards connecting monotone Hurwitz numbers to Gromov–Witten theory via combinatorial methods of **tropical geometry**.



Overview

Overview

Riemann surfaces

Recall: **Riemann surfaces** are one-dimensional complex manifolds.

Hurwitz theory is concerned with the enumeration von holomorphic maps between *compact Riemann surfaces*, i.e. finite regular morphisms between complex smooth algebraic curves.

Maps between Riemann surface

Let S_1, S_2 compact Riemann surfaces, $f\colon S_2 \to S_1$ a non-constant holomorphic map.
For all $y \in S_2$, we have that f locally at y is given by

$$z \mapsto z^{n_y},$$

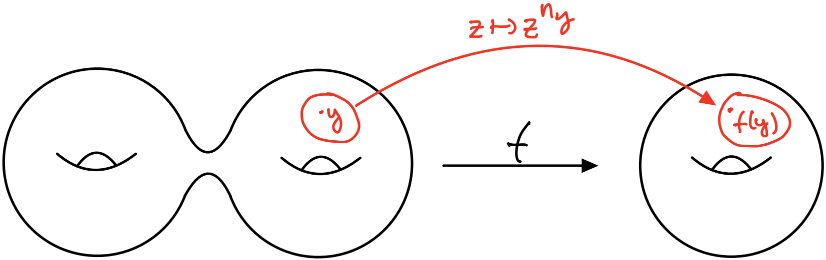
where $n_y \in \mathbb{N}_{\geq 1}$.

Maps between Riemann surface

Let S_1, S_2 compact Riemann surfaces, $f\colon S_2 \to S_1$ a non-constant holomorphic map.
For all $y \in S_2$, we have that f locally at y is given by

$$z \mapsto z^{n_y},$$

where $n_y \in \mathbb{N}_{\geq 1}$.



Maps between Riemann surface

Let S_1, S_2 compact Riemann surfaces, $f\colon S_2 \to S_1$ a non-constant holomorphic map.
For all $y \in S_2$, we have that f locally at y is given by

$$z \mapsto z^{n_y},$$

where $n_y \in \mathbb{N}_{\geq 1}$.
We call n_y the **ramification index of** y.

Maps between Riemann surface

Let S_1, S_2 compact Riemann surfaces, $f\colon S_2 \to S_1$ a non-constant holomorphic map.
For all $y \in S_2$, we have that f locally at y is given by

$$z \mapsto z^{n_y},$$

where $n_y \in \mathbb{N}_{\geq 1}$.
We call n_y the **ramification index of** y.

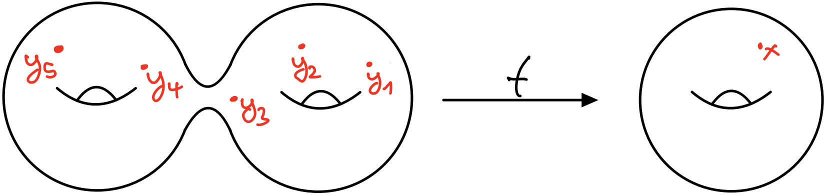
For all but finitely many y, we have $n_y = 1$.

Maps between Riemann surface

For $f : S_2 \to S_1$, let $x \in S_1$ and

$$f^{-1}(x) = \{y_1, \dots, y_s\},$$

then, we call $\mu_x = (n_{y_1}, \dots, n_{y_s})$ the **ramification profile** of x.



$\mu_x = (n_{y_1}, n_{y_2}, n_{y_3}, n_{y_4}, n_{y_5})$

Maps between Riemann surface

For $f: S_2 \to S_1$, let $x \in S_1$ and

$$f^{-1}(x) = \{y_1, \dots, y_s\},$$

then, we call $\mu_x = (n_{y_1}, \dots, n_{y_s})$ the **ramification profile** of x.

Fact

All ramification profiles μ_x are partitions of the same number d, i.e. $d = n_{y_1} + \cdots + n_{y_s}$.
We call d the **degree** $\deg(f)$ **of** f.

Maps between Riemann surface

For $f\colon S_2 \to S_1$, let $x \in S_1$ and

$$f^{-1}(x) = \{y_1, \ldots, y_s\},$$

then, we call $\mu_x = (n_{y_1}, \ldots, n_{y_s})$ the **ramification profile** of x.

Fact

All ramification profiles μ_x are partitions of the same number d, i.e. $d = n_{y_1} + \cdots + n_{y_s}$.
We call d the **degree** $\deg(f)$ **of** f.

For all but finitely many $x \in S_1$ we have $\mu_x = (1, \ldots, 1)$ and x has $\deg(f)$ preimages. We call such x **unramified**.

Maps between Riemann surface

For $f\colon S_2 \to S_1$, let $x \in S_1$ and

$$f^{-1}(x) = \{y_1, \ldots, y_s\},$$

then, we call $\mu_x = (n_{y_1}, \ldots, n_{y_s})$ the **ramification profile** of x.

Fact

All ramification profiles μ_x are partitions of the same number d, i.e. $d = n_{y_1} + \cdots + n_{y_s}$.
We call d the **degree** $\deg(f)$ **of** f.

For all but finitely many $x \in S_1$ we have $\mu_x = (1, \ldots, 1)$ and x has $\deg(f)$ preimages. We call such x **unramified**.
If $\mu_x = (2, 1 \ldots, 1)$, we call x **simply ramified**.

Hurwitz numbers

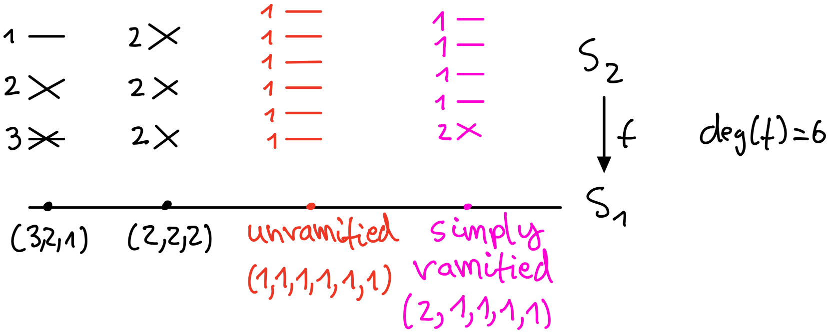
Let S_1 a compact Riemann surface.

Hurwitz numbers count holomorphic maps $f\colon S_2 \to S_1$ with prescribed ramification data.

Hurwitz numbers

Let S_1 a compact Riemann surface.

Hurwitz numbers count holomorphic maps $f\colon S_2 \to S_1$ with prescribed ramification data.



Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \ldots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \ldots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers $H_r(\mu, \nu)$ count holomorphic maps $f : S \to \mathbb{P}^1$ where

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \ldots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers $H_r(\mu, \nu)$ count holomorphic maps $f : S \to \mathbb{P}^1$ where

- S is a compact Riemann surface;

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \ldots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers $H_r(\mu, \nu)$ count holomorphic maps $f : S \to \mathbb{P}^1$ where

- S is a compact Riemann surface;
- the ramification profile of the **south pole** 0 is μ;

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \ldots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers $H_r(\mu, \nu)$ count holomorphic maps $f : S \to \mathbb{P}^1$ where

- S is a compact Riemann surface;
- the ramification profile of the **south pole** 0 is μ;
- the ramification profile of the **north pole** ∞ is ν;

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \ldots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers $H_r(\mu, \nu)$ count holomorphic maps $f : S \to \mathbb{P}^1$ where

- S is a compact Riemann surface;
- the ramification profile of the **south pole** 0 is μ;
- the ramification profile of the **north pole** ∞ is ν;
- p_i is simply ramified with $\mu_{p_i} = (2, 1, \ldots, 1)$;

Double Hurwitz numbers

Some specifications of ramification data yield Hurwitz numbers of particular interest.

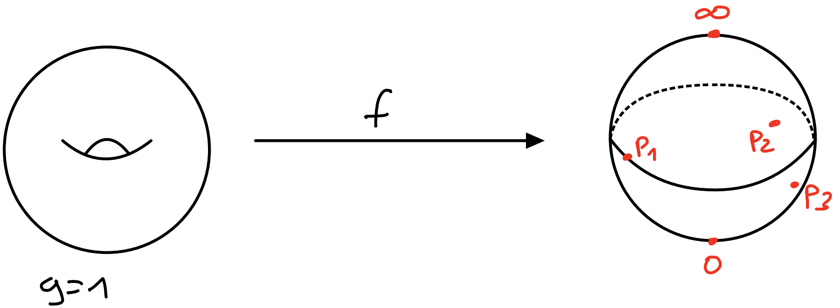
Definition

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Further let $\mathbb{P}^1$ the Riemann sphere and fix $p_1, \dots, p_r \in \mathbb{P}^1$.

Double Hurwitz numbers $H_r(\mu, \nu)$ count holomorphic maps $f : S \to \mathbb{P}^1$ where

- S is a compact Riemann surface;
- the ramification profile of the **south pole** 0 is μ;
- the ramification profile of the **north pole** ∞ is ν;
- p_i is simply ramified with $\mu_{p_i} = (2, 1, \dots, 1)$;
- all other points are unramified.

Double Hurwitz numbers



Computation via the symmetric group

In his original paper, Hurwitz proved:

- Hurwitz numbers may be computed via factorisations in the symmetric group.

Computation via the symmetric group

In his original paper, Hurwitz proved:

- Hurwitz numbers may be computed via factorisations in the symmetric group.

Notation

Let $\sigma \in S_d$ a permutation. We call the associated partition its **cycle type** $C(\sigma)$.

Computation via the symmetric group

In his original paper, Hurwitz proved:

- Hurwitz numbers may be computed via factorisations in the symmetric group.

Notation

Let $\sigma \in S_d$ a permutation. We call the associated partition its **cycle type** $C(\sigma)$.
E.g. $C((1\,2\,3)(4\,5)(6)) = (3, 2, 1)$.

Computation via the symmetric group

Theorem (Hurwitz '1892)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we have:

$$H_r(\mu,\nu) = \frac{1}{d!} \cdot \left| \left\{ \begin{array}{l} (\sigma_1, \tau_1, \ldots, \tau_r, \sigma_2): \\ \bullet\ \sigma_1, \sigma_2, \tau_i \in S_d \\ \bullet\ C(\sigma_1) = \mu,\ C(\sigma_2) = \nu \\ \bullet\ \tau_i \text{ transposition} \\ \bullet\ \sigma_1 \tau_1 \cdots \tau_r = \sigma_2 \end{array} \right\} \right|$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2,2)$.

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2, 2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$ we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2.$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2,2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$ we have

$$\sigma_1 \tau_1 \tau_2 \tau_3 = \sigma_2. \quad \checkmark$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2, 2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$ we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad \checkmark$$

Proceeding through all tuples, we obtain $H_r(\mu, \nu) = 14$.

Overview

Gromov–Witten theory

Gromov–Witten theory originates in mathematical physics

- Study of the intersection theory of parameter spaces of curves, more precisely their moduli spaces.
- Central objects are so-called **Gromov–Witten invariants**
- Revolutionised enumerative algebraic geometry

Gromov–Witten theory

Gromov–Witten theory originates in mathematical physics

- Study of the intersection theory of parameter spaces of curves, more precisely their moduli spaces.
- Central objects are so-called **Gromov–Witten invariants**
- Revolutionised enumerative algebraic geometry

Main idea: To enumerate all curves satisfying conditions $A_1, \ldots, A_n$, one observes that each A_i corresponds to a subvariety in the respective moduli space. The total number is the degree of the intersection product.

Gromov–Witten theory

Gromov–Witten theory originates in mathematical physics

- Study of the intersection theory of parameter spaces of curves, more precisely their moduli spaces.
- Central objects are so-called **Gromov–Witten invariants**
- Revolutionised enumerative algebraic geometry

Main idea: To enumerate all curves satisfying conditions $A_1, \ldots, A_n$, one observes that each A_i corresponds to a subvariety in the respective moduli space. The total number is the degree of the intersection product.

Connections between Gromov–Witten and Hurwitz theory via the celebrated ELSV formula.

ELSV formula

ELSV formula (Ekedahl, Lando, Shapiro, Vainshtein '99/'01)

Let $d > 0$, $r \geq 0$ and $\mu, \nu \vdash d$ with $\nu = (1, \ldots, 1)$. Then, we have

$$H_r(\mu, \nu) =$$
$$(\text{Comb. factor}) \cdot \left(\text{Gromov–Witten invariants of } \overline{M}_{g,n}\right).$$

ELSV formula

ELSV formula (Ekedahl, Lando, Shapiro, Vainshtein '99/'01)

Let $d > 0$, $r \geq 0$ and $\mu, \nu \vdash d$ with $\nu = (1, \ldots, 1)$. Then, we have

$$H_r(\mu, \nu) = \\ (\text{Comb. factor}) \cdot (\text{Gromov–Witten invariants of } \overline{M}_{g,n}) .$$

In other words: **Hurwitz numbers enumerate certain Gromov–Witten invariants!**

Applications of the ELSV formula

The ELSV formula implies various important theorems on the intersection theory of $\overline{M}_{g,n}$.

- Witten's conjecture/Kontsevich's theorem
- Virasoro constraints
- Faber's λ_g conjecture

Polynomiality of Hurwitz numbers

It also allows applications in Hurwitz theory

E.g. it implies the Goulden–Jackson conjecture ’99.

Conjecture (Goulden, Jackson ’99)

Let $d > 0$, $r \geq 0$ and $\mu, \nu \vdash d$ with $\nu = (1, \ldots, 1)$. Then, we have

$$H_r(\mu, \nu) = (\mathrm{Comb.\ factor}) \cdot P_g(\mu),$$

where $P_g(\mu)$ is a polynomial in the entries of μ.

Generalisations

Question

What about other ν?

Generalisations

Question

What about other ν?

2005: Goulden, Jackson and Vakil propose a programme towards an ELSV type formula for all double Hurwitz numbers.

Generalisations

Question

What about other ν?

2005: Goulden, Jackson and Vakil propose a programme towards an ELSV type formula for all double Hurwitz numbers.

Idea: Derive polynomial behaviour of $H_r(\mu, \nu)$ for any ν and reconstruct an ELSV type formula.

Theorem of Goulden, Jackson and Vakil

Let $m, n \in \mathbb{N}_{\geq 1}$ and consider

$$\mathcal{H}_{m,n} := \{(\mu, \nu) \in \mathbb{N}^m \times \mathbb{N}^n \mid \sum \mu_i = \sum \nu_i\}.$$

Theorem of Goulden, Jackson and Vakil

Let $m, n \in \mathbb{N}_{\geq 1}$ and consider

$$\mathcal{H}_{m,n} := \{(\mu, \nu) \in \mathbb{N}^m \times \mathbb{N}^n \mid \sum \mu_i = \sum \nu_i\}.$$

Theorem (Goulden, Jackson, Vakil '05)

Let $r \geq 0$. The map

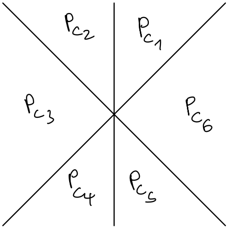
$$\begin{aligned} \mathcal{H}_{m,n} &\to \mathbb{Q} \\ (\mu, \nu) &\mapsto H_r(\mu, \nu). \end{aligned}$$

is **piecewise polynomial**.

Theorem of Goulden, Jackson and Vakil

Hyperplane arrangement in $\mathcal{H}_{m,n}$.

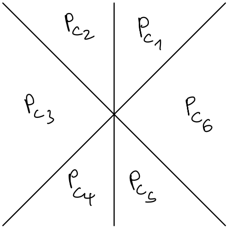
- Polynomial in every maximal cell.



Theorem of Goulden, Jackson and Vakil

Hyperplane arrangement in $\mathcal{H}_{m,n}$.

- Polynomial in every maximal cell.



Conjecture (Goulden, Jackson, Vakil '05; Bayer, Cavalieri, Johnson, Markwig '12)

Concrete proposal for an ELSV type formula for double Hurwitz numbers

Polynomiality of double Hurwitz numbers

Question (Shadrin, Shapiro, Vainshstein '06)

What is the difference $P_{C_1} - P_{C_2}$?

Polynomiality of double Hurwitz numbers

Question (Shadrin, Shapiro, Vainshstein '06)

What is the difference $P_{C_1} - P_{C_2}$?

Theorem (Cavalieri, Johnson, Markwig '10)

The difference $P_{C_1}(\mu, \nu) - P_{C_2}(\mu, \nu)$ may be expressed recursively in terms of Hurwitz numbers with smaller input data. (**Recursive wall-crossing formula**)

Polynomiality of double Hurwitz numbers

Question (Shadrin, Shapiro, Vainshstein '06)

What is the difference $P_{C_1} - P_{C_2}$?

Theorem (Cavalieri, Johnson, Markwig '10)

The difference $P_{C_1}(\mu,\nu) - P_{C_2}(\mu,\nu)$ may be expressed recursively in terms of Hurwitz numbers with smaller input data. (**Recursive wall-crossing formula**)

Since 2010: Strong progress, but Goulden–Jackson–Vakil/ Bayer–Cavalieri–Johnson–Markwig conjecture remains open.

Overview

Harish-Chandra–Itzykson–Zuber integral

Harish-Chandra–Itzykson–Zuber integral: Central object in random matrix theory.

$$\int_{U(N)} e^{zN\mathrm{Tr}(AUBU^{-1})}\mathrm{d}U$$

Harish-Chandra–Itzykson–Zuber integral

Harish-Chandra–Itzykson–Zuber integral: Central object in random matrix theory.

$$\int_{U(N)} e^{zN\mathrm{Tr}(AUBU^{-1})}\mathrm{d}U$$

Theorem (Goulden, Guay-Paquet, Novak '11)

The coefficients in the Taylor expansion of the HCIZ integral are **monotone** double Hurwitz numbers.

Harish-Chandra–Itzykson–Zuber integral

Harish-Chandra–Itzykson–Zuber integral: Central object in random matrix theory.

$$\int_{U(N)} e^{zN\mathrm{Tr}(AUBU^{-1})}\mathrm{d}U$$

Theorem (Goulden, Guay-Paquet, Novak '11)

The coefficients in the Taylor expansion of the HCIZ integral are **monotone** double Hurwitz numbers.

(**Motivation:** Conjecture about the convergence of the HCIZ integral – proved in 2020 by Novak.)

Monotone double Hurwitz numbers

Definition (Goulden, Guay-Paquet, Novak '11)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we define **monotone double Hurwitz numbers**:

$$\vec{H}_r(\mu,\nu) = \frac{1}{d!} \cdot \left| \left\{ \begin{array}{l} (\sigma_1, \tau_1, \ldots, \tau_r, \sigma_2): \\ \bullet\ \sigma_1, \sigma_2, \tau_i \in S_d \\ \bullet\ C(\sigma_1) = \mu,\ C(\sigma_2) = \nu \\ \bullet\ \tau_i \text{ Transposition} \\ \bullet\ \sigma_1 \tau_1 \cdots \tau_r = \sigma_2 \\ \bullet\ \tau_i = (r_i\, s_i) \text{ mit } r_i < s_i, \text{ then: } s_i \leq s_{i+1} \end{array} \right\} \right|$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2, 2)$.

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2,2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$, we have

$$\sigma_1 \tau_1 \tau_2 \tau_3 = \sigma_2.$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2,2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad ⚡$$

Example

Let $d = 4,\ r = 3,\ \mu = (4),\ \nu = (2,2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad \lightning$$

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,2), \tau_2 = (1\,3), \tau_3 = (2\,3)$, $\sigma_2 = (1\,4)(2\,3)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2.$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2,2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad \lightning$$

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,2), \tau_2 = (1\,3), \tau_3 = (2\,3)$, $\sigma_2 = (1\,4)(2\,3)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad \checkmark$$

Example

Let $d = 4$, $r = 3$, $\mu = (4)$, $\nu = (2,2)$.

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,4), \tau_2 = (2\,3), \tau_3 = (1\,3)$, $\sigma_2 = (1\,3)(2\,4)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad ↯$$

- For $\sigma_1 = (1\,2\,3\,4), \tau_1 = (1\,2), \tau_2 = (1\,3), \tau_3 = (2\,3)$, $\sigma_2 = (1\,4)(2\,3)$, we have

$$\sigma_1\tau_1\tau_2\tau_3 = \sigma_2. \quad \checkmark$$

Proceedings through all tuples, we obtain $\vec{H}_r(\mu,\nu) = \frac{25}{4}$.

Goal

Question

Is there a connection between monotone Hurwitz numbers and Gromov–Witten theory?

Goal

Question

Is there a connection between monotone Hurwitz numbers and Gromov–Witten theory?

1. Our ansatz is via **tropical geometry**.

Goal

Question

Is there a connection between monotone Hurwitz numbers and Gromov–Witten theory?

1. Our ansatz is via **tropical geometry**.
2. Long-term goal: **ELSV-Typ formula** for monotone double Hurwitz numbers.

Overview

Tropical geometry

Tropical geometry: Conceptualisation of a combinatorial approach to algebraic geometry

- Inception in 2000s through work of Mikhalkin and Sturmfels following a suggestion of Kontsevich
- **Tropicalisation:** Transformation of algebro–geometric to combinatorial objects

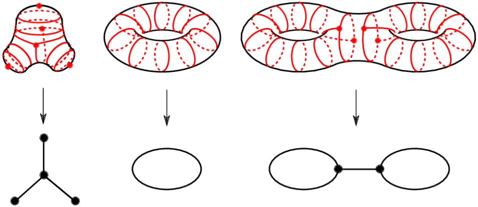


Figure: Colla, Marelli - "Pair of pants decomposition of 4-manifolds"

Tropical Hurwitz numbers

Cavalieri, Johnson, Markwig '08: Graph-theoretic interpretation of classical double Hurwitz numbers.

Tropical Hurwitz numbers

Cavalieri, Johnson, Markwig '08: Graph-theoretic interpretation of classical double Hurwitz numbers. $\longrightarrow$ Central role in proof of recursive wall-crossing formulae '10.

Tropical Hurwitz numbers

Cavalieri, Johnson, Markwig '08: Graph-theoretic interpretation of classical double Hurwitz numbers. $\longrightarrow$ Central role in proof of recursive wall-crossing formulae '10.

Tropical monotone Hurwitz numbers

Problem: No geometric interpretation of monotone double Hurwitz numbers.

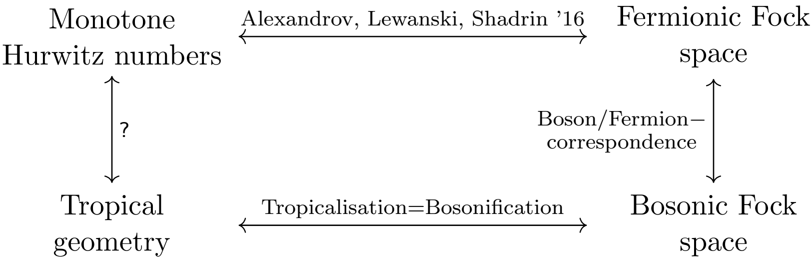
- Monotone factorisations not preserved under conjugation.

Tropical monotone Hurwitz numbers

Problem: No geometric interpretation of monotone double Hurwitz numbers.

- Monotone factorisations not preserved under conjugation.

Alternative approach: Fermionic/bosonic Fock space



Tropical monotone Hurwitz numbers

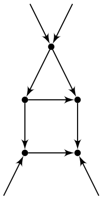
Theorem (H. '17; H., Lewanski '18)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we have

$$\vec{H}_r(\mu,\nu) = \sum_{\Gamma}$$

where

- Γ finitely many oriented graphs



Tropical monotone Hurwitz numbers

Theorem (H. '17; H., Lewanski '18)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we have

$$\vec{H}_r(\mu, \nu) = \sum_{\Gamma} \mathrm{GW}(\Gamma),$$

where

- Γ finitely many oriented graphs
- $\mathrm{GW}(\Gamma)$ is given by **relative Gromov–Witten invariants**

Tropical monotone Hurwitz numbers

Theorem (H. '17; H., Lewanski '18)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we have

$$\vec{H}_r(\mu,\nu) = \sum_{\Gamma} \mathrm{GW}(\Gamma),$$

where

- Γ finitely many oriented graphs
- $\mathrm{GW}(\Gamma)$ is given by **relative Gromov–Witten invariants**

In other words: **Relative Gromov–Witten invariants compute monotone double Hurwitz numbers!**

Overview

ELSV-Typ formula for $\vec{H}_r(\mu, \nu)$

An ELSV-Typ formula for monotone double Hurwitz numbers is an open problem.

ELSV-Typ formula for $\vec{H}_r(\mu, \nu)$

An ELSV-Typ formula for monotone double Hurwitz numbers is an open problem.

Idea: Follow the approach of Goulden, Jackson and Vakil via polynomiality.

ELSV-Typ formula for $\vec{H}_r(\mu, \nu)$

An ELSV-Typ formula for monotone double Hurwitz numbers is an open problem.

Idea: Follow the approach of Goulden, Jackson and Vakil via polynomiality.

Theorem (Goulden, Guay-Paquet, Novak '14)

Monotone double Hurwitz numbers are piecewise polynomial with the same hyperplane arrangement as their classical counterpart.

ELSV-Typ formula for $\vec{H}_r(\mu, \nu)$

An ELSV-Typ formula for monotone double Hurwitz numbers is an open problem.

Idea: Follow the approach of Goulden, Jackson and Vakil via polynomiality.

Theorem (Goulden, Guay-Paquet, Novak '14)

Monotone double Hurwitz numbers are piecewise polynomial with the same hyperplane arrangement as their classical counterpart.

- Purely an existence theorem;

ELSV-Typ formula for $\vec{H}_r(\mu, \nu)$

An ELSV-Typ formula for monotone double Hurwitz numbers is an open problem.

Idea: Follow the approach of Goulden, Jackson and Vakil via polynomiality.

Theorem (Goulden, Guay-Paquet, Novak '14)

Monotone double Hurwitz numbers are piecewise polynomial with the same hyperplane arrangement as their classical counterpart.

- Purely an existence theorem;
- Wall-crossing formulae not approachable with the involved methods $\longrightarrow$ **Open problem**

Wall-Crossing formulae for $\vec{H}_r(\mu, \nu)$

Using the tropical interpretation for monotone double Hurwitz numbers, we could prove the following.

Theorem (H. '17; H., Kramer, Lewanksi '17; H., Lewanski '19)

- Algorithms to compute monotone double Hurwitz numbers
- Monotone double Hurwitz numbers admit recursive wall-crossing formulae.

Wall-Crossing formulae for $\vec{H}_r(\mu, \nu)$

Sketch of proof

- Use the tropical interpretation $\vec{H}_r(\mu, \nu) = \sum_\Gamma \mathrm{GW}(\Gamma)$
- Observe: $\mathrm{GW}(\Gamma)$ is a discrete integral over a polytope $\longrightarrow$ Polynomiality via Ehrhart theory
- Combinatorial analysis of the structure of the polytopes in different maximal cells of the hyperplane arrangement yields wall-crossing formulae.

$\square$

Wall-Crossing formulae for $\vec{H}_r(\mu, \nu)$

Sketch of proof

- Use the tropical interpretation $\vec{H}_r(\mu, \nu) = \sum_\Gamma \mathrm{GW}(\Gamma)$
- Observe: $\mathrm{GW}(\Gamma)$ is a discrete integral over a polytope $\longrightarrow$ Polynomiality via Ehrhart theory
- Combinatorial analysis of the structure of the polytopes in different maximal cells of the hyperplane arrangement yields wall-crossing formulae.

$\square$

Our results also hold for **strictly monotone Hurwitz numbers** ($s_i < s_{i+1}$), that are equivalent to an enumeration of Grothendieck dessins d'enfants.

Overview

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Definition

Let $\mathbb{T}$ be the torus, fix $d > 0$, $r \geq 0$ and $p_1, \ldots, p_r \in \mathbb{T}$.

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Definition

Let $\mathbb{T}$ be the torus, fix $d > 0$, $r \geq 0$ and $p_1, \ldots, p_r \in \mathbb{T}$. Then elliptic Hurwitz numbers $N_{d,r}$ enumerate holomorphic maps $f : S \to \mathbb{T}$, where

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Definition

Let $\mathbb{T}$ be the torus, fix $d > 0$, $r \geq 0$ and $p_1, \ldots, p_r \in \mathbb{T}$. Then elliptic Hurwitz numbers $N_{d,r}$ enumerate holomorphic maps $f : S \to \mathbb{T}$, where

- S is a compact Riemann surface;

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Definition

Let $\mathbb{T}$ be the torus, fix $d > 0$, $r \geq 0$ and $p_1, \dots, p_r \in \mathbb{T}$. Then elliptic Hurwitz numbers $N_{d,r}$ enumerate holomorphic maps $f : S \to \mathbb{T}$, where

- S is a compact Riemann surface;
- p_i is simply ramified with $\mu_{p_i} = (2, 1, \dots, 1)$;

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Definition

Let $\mathbb{T}$ be the torus, fix $d > 0$, $r \geq 0$ and $p_1, \dots, p_r \in \mathbb{T}$. Then elliptic Hurwitz numbers $N_{d,r}$ enumerate holomorphic maps $f : S \to \mathbb{T}$, where

- S is a compact Riemann surface;
- p_i is simply ramified with $\mu_{p_i} = (2, 1, \dots, 1)$;
- all other points are unramified.

Elliptic Hurwitz numbers

Elliptic Hurwitz numbers are deeply intertwined with the mirror symmetry conjecture.

Definition

Let $\mathbb{T}$ be the torus, fix $d > 0$, $r \geq 0$ and $p_1, \ldots, p_r \in \mathbb{T}$. Then elliptic Hurwitz numbers $N_{d,r}$ enumerate holomorphic maps $f : S \to \mathbb{T}$, where

Dijkgraaf '95: Confirms two predictions of mirror symmetry regarding the structure of elliptic Hurwitz numbers.

Quasimodularity

One of the predictions is the following:

Theorem (Dijkgraaf '95)

Let E_2, E_4, E_6 be the Ehrhart series given by

$$E_k(q) = -\frac{B_{2k}}{2k} + \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} m^{k-1} q^{mr}.$$

Then, for fixed $r \geq 2$, we have

$$\sum_{d=1}^{\infty} N_{d,r} q^d \in \mathbb{Q}[E_2, E_4, E_6],$$

i.e. it is a **quasimodular form**.

Quasimodularity

Eskin–Okounkov ’99: This quasimodularity property allows to study the asymptotics of elliptic Hurwitz numbers as d approaches infinity.

Quasimodularity

Elliptic Hurwitz numbers equivalently enumerate factorisations $(\tau_1, \ldots, \tau_r, \alpha, \beta)$, with τ_i transpositions, α, β any permutations and

$$\tau_r \cdots \tau_1 = \alpha\beta\alpha^{-1}\beta^{-1}.$$

Quasimodularity

Elliptic Hurwitz numbers equivalently enumerate factorisations $(\tau_1, \ldots, \tau_r, \alpha, \beta)$, with τ_i transpositions, α, β any permutations and

$$\tau_r \cdots \tau_1 = \alpha\beta\alpha^{-1}\beta^{-1}.$$

Analogously to double Hurwitz numbers, this gives rise to a notion of **monotone elliptic Hurwitz numbers** $\vec{N}_{d,r}$.

Quasimodularity

Elliptic Hurwitz numbers equivalently enumerate factorisations $(\tau_1, \ldots, \tau_r, \alpha, \beta)$, with τ_i transpositions, α, β any permutations and

$$\tau_r \cdots \tau_1 = \alpha\beta\alpha^{-1}\beta^{-1}.$$

Analogously to double Hurwitz numbers, this gives rise to a notion of **monotone elliptic Hurwitz numbers** $\vec{N}_{d,r}$.

Quasimodularity

Theorem (H.–van Ittersum–Leid '19)

Elliptic monotone Hurwitz numbers admit a tropical interpretation.

Quasimodularity

Theorem (H.–van Ittersum–Leid '19)

Elliptic monotone Hurwitz numbers admit a tropical interpretation.

From this interpretation, it follows that for fixed $r \geq 2$, the series

$$\sum_{d=1}^{\infty} \vec{N}_{d,r} q^d$$

is a quasimodular form as well.