Harder-Narasimhan theory for gauged maps
(joint with Dan Halpern-Leistner)

<u>Moduli of curves:</u>

Classification of smooth projective connected curves.

- Topologically they classified by genus g

X_g

<u>Q</u>: How many complex structures (up to iso) can we put on X_g?

<u>A</u>: For $g \geq 1$, there are infinitely many.

<u>Riemann's moduli space</u>: For $g \geq 2$, there is a complex variety (orbifold) M_g parametrizing complex structures on X_g.

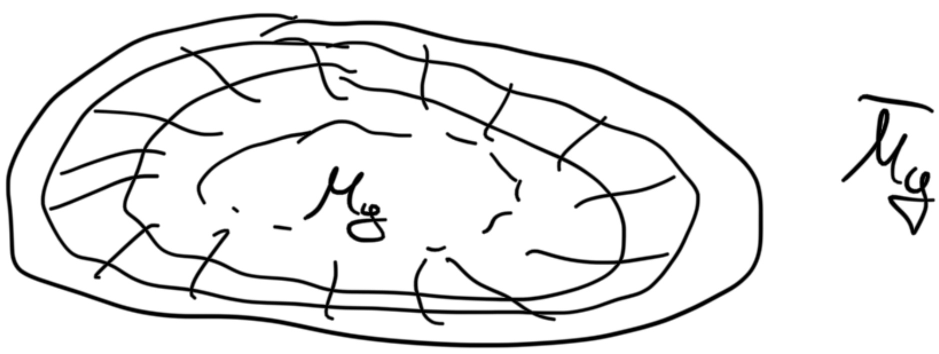
$$\dim_{\mathbb{C}}(M_g) = 3g-3.$$

<u>Techniques</u>: (Deligne - Mumford)

① $M_g \subset \overline{M_g}$ = stable curves

(Compactification)

② The boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ they can be understood in terms of curves of lower genus.



- How to count?

- Computing a volume : $\int_{\overline{\mathcal{M}}_g} \omega$ "count"

- K-theoretic counts:

$$\sum_i n_i [E_i] \longmapsto \sum_i n_i \chi(E_i)$$

$[E_i]$: vector bundles

<u>Gromov-Witten</u>: X projective variety

$$\mathcal{M}_g(X)_\beta \subset \overline{\mathcal{M}}_g(X)_\beta$$

$\mathcal{M}_g(X)_\beta$ = maps from genus g smooth curves

$\overline{\mathcal{M}}_g(X)_\beta$ = stable maps.

<u>Today</u>: gauged maps from Riemann surfaces

("count curves on stacks").

Set up: X = affine variety
$\circlearrowleft$
G = reductive group (finite, GL_n, SL_n, SO_n).

C = fixed smooth connected projective curve.

Moduli $\quad Map(C, X/G)$

parametrizes: pairs (E, s)

- E is a G-bundle on C
- $s : C \to E(X)$

$$\begin{tikzcd} E(X) = (X \times E)/G \arrow[d, shift left] \\ C \arrow[u, shift left, "s"] \end{tikzcd}$$

Examples 1 $\quad X = pt, \; G = GL_n$

$$Map(C, pt/GL_n) = Map(C, BGL_n) = Bun_n(C)$$

Topological invariant: degree of vector bundle d.

Problem: there is no complex projective variety that classifies all vector bundles.

Solution 1: work with the stack $Bun_n(C)_d$

Solution 2: (Mumford) there is a class of "sufficiently rigid" vector bundles called semistable vector bundles. They are classified by a projective moduli space.

- How to count?

(Finite fields): Harder - Narasimhan

they construct a stratification of boundary.

(Complex numbers:) There is a ~~nice~~ line bundle $\mathcal{L}$ (determinant line bundle) on $M^{ss}_{n,d}$

(Verlinde formula):

$$\dim\left(H^0(\mathcal{L}^{\otimes k})\right) = \left(\frac{k}{g}\right)^g \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{(-1)^{id}}{\left(\sin\left(\frac{(i+1)\pi}{k+2}\right)\right)^{2g-2}}$$

$\boxed{\text{Example 2}}$ $g=1$, $X = \mathfrak{g} = \mathrm{Lie}(G) \underset{Ad}{\circlearrowleft} G$

$$\mathrm{Map}(C, \mathfrak{g}/G) = \left\{ \begin{array}{l} \cdot\ E \ \ G\text{-bundle on } C \\ \cdot\ s \in H^0(\mathrm{Ad}(E)) := H^0(E(\mathfrak{g})). \end{array} \right\}$$

Higgs bundles

- $\mathfrak{g}/G \longrightarrow \mathfrak{g}/\!/G = \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$

$$\mathbb{G}_m \curvearrowright \mathrm{Map}(C, \mathfrak{g}/G)^{ss} \subset \mathrm{Map}(C, \mathfrak{g}/G) \longrightarrow \mathrm{Map}(C, \mathfrak{g}/\!/G) = \mathfrak{g}/\!/G \curvearrowleft \mathbb{G}_m$$

$\mathbb{G}_m = \mathbb{C}^\times$

Hitchin fibration.

$\cdot\ M_{Pol}$ - -

Idea: K-theory class that is $\mathbb{G}_m$-equivariant

pushforward to get a "graded" K-theory class on $\mathfrak{g}/\!/G$.

The index of each grading is finite

$$\sum_n x_n q^n = f(q)$$

Verlinde formula for Higgs bundles: $\begin{cases} \text{Andersen-Gukov-Pei} \\ \text{Halpern-Leistner} \end{cases}$

- Let's go back to general $\mathrm{Map}(C, X/G)$

$$\mathrm{Map}(C, X/G) \longrightarrow \mathrm{Map}(C, X/\!/G) = X/\!/G$$

- Let's take $\chi : G \longrightarrow \mathbb{C}^\times$

(this roughly a stability condition in the sense of GIT $X \circlearrowleft G$)

$\boxed{\text{THM}}$ (Halpern-Leistner - H) For any X, G, χ

①
$$\begin{array}{ccc} \mathrm{Map}(C, X/G)_d^{\chi\text{-ss}} & \subset & \mathrm{Map}(C, X/G)_d \\ \downarrow & & \downarrow \\ M & \xrightarrow{\text{proper}} & X/\!/G \end{array}$$

moduli space

② There is a Harder-Narasimhan stratification of the unstable locus.

(3) (Verlinde formula) Assume X is a linear representation. Then there is an explicit formula for the indexes of Atiyah–Bott classes (in the graded sense).

- <u>Ideas of proof</u>:

(1) Wall crossing $\quad X \subset \overline{X} \circlearrowleft G$

$$\begin{tikzcd} \underbrace{\mathrm{Map}(C, X/G)} \subset \overline{\mathcal{M}}_C(\overline{X}/G) \arrow[d, "\text{proper}"] \\ \mathrm{Bun}_G(C) \end{tikzcd}$$

Extra stability parameter χ, ξ for $\overline{\mathcal{M}}_C(\overline{X}/G)$

Because of properness, we can understand $\overline{\mathcal{M}}_C(\overline{X}/G)$ and relate it to HN of $\mathrm{Bun}_G(C)$.

We take $\xi \to \infty$.

(2) Infinite dimensional GIT.

check implicit criteria for existence of moduli spaces using uniformization by affine Grassmannian?

$$
\begin{tikzcd}
\mathrm{Map}(\cancel{C}, X/G) & \mathrm{Map}(C, X^{\lambda\geq 0}/P_\lambda) \arrow[l] \arrow[d] \\
& \mathrm{Map}(C, X^{\lambda=0}/L_\lambda)^{\lambda\text{-ss}}
\end{tikzcd}
$$

$\lambda : \mathbb{G}_m \to G$