Mirror Symmetry & Lagrangian Torus Fibrations

Tuesday, April 5, 2022 5:14 PM

- Mirror symmetry is a conjectured relation between
  - "A-side" sympletic geometry
  - "B-side" algebraic geometry

on a pair of "mirror spaces"

$$\begin{array}{ccc} X^A & & X^B \\ & \searrow \quad \swarrow & \\ & Q & \end{array}$$

A-side

Def^n A sympletic mfld 2n-$\mathbb{R}$ dimension manifold X^A with a 2-form $\omega \in \Omega^2(X^A, \mathbb{R})$ st.

- $d\omega = 0$ and
- $\omega^n \neq 0$

- Very flexible (no local invariants)
- Local model $\mathbb{R}^{2n}$, $\sum_{i=1}^{n} dx_i \wedge dy_i$

$\{x_1 \ldots x_n, y_1 \ldots y_n\}$

Example: Cotangent Bundle

Let M be a manifold w/ coordinates $q_1 \ldots q_n$

Then T^*M has local coord. $(q_1 \cdots q_n), (p_1 \cdots p_n)$

coordinates in the dq_i dir.

Then $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$ is symplectic form.

ω independent on local coord.

Lagrangian Submanifolds

Def^n A submanifold $L \subset X^A$ is Lagrangian if

- $\dim(L) = n = \frac{1}{2}\dim X^A$
- $\omega|_L = 0$.

(note that n is the largest dimension on which this can occur).

- Example: $\mathbb{R}^n \subset \mathbb{R}^{2n}$.
- Fact: Every Lag. submanifold in X^A has neighborhood like a portion of T^*L

B-side (Complex-Geometry)

Def^n An almost-complex manifold is a 2n dimensional manifold X^B with a bundle isomorphism

$$J : TX^A \to TX^A$$

st. $J^2 = -\mathrm{id}$.

Examples: If X^A is complex then

- i gives an ACS on X^A.

Example: Tangent bundle w/ connection

Given Q a manifold with connection ∇ on TQ we obtain an ACS on a 2n manifold

$$J : T(TQ) \to T(TQ)$$

which swaps the horizontal & vertical subbundles of $T(TQ)$.

In local coordinates: $TQ = (q_1 \cdots q_n, v_1 \cdots v_n)$

$J\partial_{q_i} = \partial_{v_i} \qquad J\partial_{v_i} = -\partial_{q_i}$

Almost complex Submanifolds

An almost complex submanifold of X^B is $Y^B \subset X^B$, (Y^B, J_Y) also almost complex.

$$J_X|_{Y^B} = J_Y.$$

Example If $V \subset Q$ a submanifold, then

$$TV \subset TQ$$

is an almost complex submanifold.

Affine geometry

Q is called Affine if there exists a lattice

$$T_{\mathbb{Z}}Q \subset TQ$$

(also gives a dual lattice $T^*_{\mathbb{Z}}Q \subset T^*Q$)

Example: $\mathbb{R}^n$, $\mathbb{R}^n/\mathbb{Z}^n$

Affine submanifold: $V \subset Q$ st.

$T_{\mathbb{Z}}Q|_V$ is a lattice on TV.

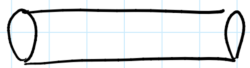
(lines/planes/... of rational slope).

Affine to A-side

Given Q affine, let

$$X^A := T^*Q/T^*_{\mathbb{Z}}Q.$$

Example: $Q = \mathbb{R}^1$



$X^A = T^*\mathbb{R}/T_{\mathbb{Z}}\mathbb{R}$

Example: $Q = \mathbb{R}^n$

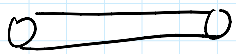
$X^A = T^*T^n = "(\mathbb{C}^*)^n"$.

Affine to B-side.

Given Q affine let

$$X^B = TQ/T_{\mathbb{Z}}Q.$$

Example: $Q = \mathbb{R}^1$



$X^B = TQ/T_{\mathbb{Z}}Q$

$\rightsquigarrow$ comes w/ complex structure identifying it with $(\mathbb{C}^*)$.

Given $V \subset Q$ affine submanifold

$$N^*V/N^*_{\mathbb{Z}}V \subset T^*Q/T^*_{\mathbb{Z}}Q \qquad\qquad TV/T_{\mathbb{Z}}V \subset TQ/T_{\mathbb{Z}}Q$$

Lagrangian Submanifold $\qquad\qquad$ Almost complex Submanifold.

Conjecture: There is a relation between the

- Lagrangian Submanifolds of X^A
- AC Submanifolds (coherent sheaves) on X^B
- ~~Affine~~ Tropical submanifolds of Q