# The minimal projective bundle dimension and toric $2$-Fano manifolds

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May 25, 2023

A complex projective manifold $X$ is *Fano* if $-K_X$ is ample, or equivalently if the the first Chern class $c_1(T_X)$ is positive.

Ex: $\mathbb{P}^n$, smooth complete intersections of low deg in $\mathbb{P}^n$, rat'l homog.

**Special Properties of Fanos**

**Theorem: (Mori, 1979)**

Any Fano manifold is covered by rational curves.

**Theorem: (Campana, Kollár-Miyaoka-Mori, 1992)**

Any Fano manifold is rationally connected.


deJong-Starr (2006): introduced & investigated possible conditions for higher rat'l connectedness

# Higher Fano Manifolds

**Definition: $k$-Fano**

A smooth projective variety $X$ is $2$-*Fano* if it is Fano and its second Chern character $ch_2(X) = \frac{1}{2}c_1(T_X)^2 - c_2(T_X)$ is positive, i.e., $ch_2(X) \cdot S > 0$ for every surface $S \subset X$.

In a similar way, one can define $k$-Fano varieties for any $k \geq 2$.

- $\mathbb{P}^n$ is $n$-Fano, and it is conjectured that it is the only $n$-dimensional $n$-Fano manifold.
- The geometry of higher Fano manifolds has been fairly investigated:
  - $2$-Fano manifolds + mild assumptions are covered by rational surfaces (de Jong-Starr)
  - similar results hold for higher Fano manifolds (Suzuki), (Nagaoka)

$X$ Fano + $ch_2(X).S > 0$ $\forall$ $S \subset X$ surf.

2-Fano:

- Araujo-Castravet give a classification of $2$-Fano manifolds of high index – largest integer div of $-K_X$ in Pic $X$
- (ABCJMMTV, 2022) gives a classification of homogeneous $2$-Fano manifolds
- All known examples of $2$-Fano manifolds have Picard number $1$ and relatively large index

$k$-Fano:

- Very few examples of higher Fano manifolds are known
- (ABCJMMTV, 2022) look at $3$-Fano manifolds

$\dim X = n \geq 3$, index $i_X \geq n-2$

3 Fano: $\mathbb{P}^n$

- complete int in proj space
- complete int in weighted proj space

# Toric case

Projective spaces are the only projective toric manifolds with $\rho(X) = 1$

- a classification of toric $2$-Fano manifolds could either
  1. provide the first examples of $2$-Fano manifolds with higher Picard number,
  2. or it could be an evidence that every $2$-Fano manifold has $\rho(X) = 1$.

Geometric properties of a toric variety can often be checked in the combinatorics of the associated fan

- This bridge has been exploited in search of new examples of toric $2$-Fano manifolds
- A complete (computer aided) classification is only known up to dimension $8$ (Nobili) (Sano-Sato-Suyama), and projective spaces remain the only known examples of toric $2$-Fano manifolds.

very explicit - construct a surface $S \subset X$ with $ch_2(X).S \leq 0$

# Toric case

**Conjecture**

The only toric 2-Fano manifolds are projective spaces.

**Idea**: investigate $2$-Fano manifolds by studying their *minimal dominating families of rational curves*.

**Set up**: $X =$ *smooth* and *proper* toric variety; $X \leftrightarrow \Sigma_X =$ fan
$G(\Sigma_X) =$ the set of primitive generators of one-dimensional cones

Given a cone $\sigma \in \Sigma_X$, $\sigma = \langle y_1, \ldots, y_k \rangle$
then $G(\sigma) := \{y_1, \ldots, y_k\}$

# Primitive Collections & Primitive Relations

$P = \{x_1, x_2, \ldots, x_h\} \subseteq G(\Sigma_X)$ is a *primitive collection* if $\langle x_1, x_2, \ldots, x_h \rangle \notin \Sigma_X$ but $\langle x_1, \ldots, \hat{x}_i, \ldots x_h \rangle \in \Sigma_X$, $1 \leq i \leq h$;

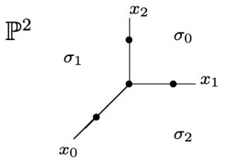
PC

Let $\sigma_P = \langle y_1, \ldots, y_k \rangle$ be minimal cone such that $x_1 + \cdots + x_h \in \sigma_P$, then there is a *primitive relation*

$$x_1 + \cdots + x_h - (a_1 y_1 + \cdots + a_k y_k) = 0 \qquad a_i > 0$$

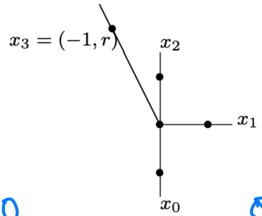
Any smooth toric variety $X$ of dim $n$ has at least one primitive relation of the form $x_1 + x_2 + \cdots + x_k = 0$, for some $2 \leq k \leq n+1$

**Ex:** $\mathbb{P}^2$




$PC = \{x_0, x_1, x_2\}$ $\quad x_0 + x_1 + x_2 = 0$

Hirzebruch Surface $\mathbb{F}_r$




$PC = \{x_0, x_2\}$

$x_0 + x_2 = 0$

$P = \{x_1, x_3\}$

$x_1 + x_3 = r x_2$

$\sigma_P = x_2$

$x_1 + x_3 - r x_2 = 0$

# Centrally Symmetric Primitive Relations

(Chen-Fu-Hwang): Minimal dominating families of rational curves on a smooth projective toric variety $X$ correspond to *primitive relations* of the form

$$x_0 + \cdots + x_m = 0, \qquad (*)$$

these primitive relations are called *centrally symmetric of order* $m+1$.

Centrally symmetric primitive collections of order $m+1$

$$\updownarrow$$

Open dense $T$-invariant $U \subset X$ and $\mathbb{P}^m$-bundle $U \to W$

CFH wanted U "small"

- Want U as big as possible.

$x_0 + x_1 + \dots + x_m = 0$

Given $P = \{x_0, x_1, \ldots, x_m\}$, a centrally symmetric primitive collection on $X$,

$$\mathcal{E}_P := \{\sigma \in \Sigma_X \mid P \cap G(\sigma) = \varnothing \text{ and } \exists P' \subsetneq P \text{ such that } P' \cup G(\sigma) \in \mathrm{PC}(X)\}$$

$$V(\mathcal{E}_P) := \bigcup_{\sigma \in \mathcal{E}_P} V(\sigma) \subset X$$

**Proposition (ABCJMMV):**

$U = X \setminus V(\mathcal{E}_P)$ admits a $\mathbb{P}^m$-bundle structure over a smooth toric variety.

$$\begin{array}{ccc} X & \supset & U \\ & & \big\downarrow \mathbb{P}^m \\ & & W \end{array}$$

$W$ = sm toric var

**Defn:** *minimal projective bundle dimension* of $X$ (*minimal* $\mathbb{P}$*-dimension*)

$$m(X) = \min_{m \in \mathbb{Z}_{>0}} \{\exists \text{ a prim relation } x_0 + \cdots + x_m = 0\} \in \{1, \ldots, \dim X\}$$

| $\dim(X)$ | # Fanos | $\#(m{=}1)$ | $\#(m{=}2)$ | $\#(m{=}3)$ | $\#(m{=}4)$ | $\#(m{=}5)$ | $\#(m{=}6)$ |
|---|---|---|---|---|---|---|---|
| 4 | 124 | 107 | 15 | 1 | 1 $\mathbb{P}^4$ | | |
| 5 | 866 | 744 | 112 | 8 | 1 | 1 $\mathbb{P}^5$ | |
| 6 | 7622 | 6333 | 1174 | 105 | 8 | 1 | 1 $\mathbb{P}^6$ |

Blowup of $\mathbb{P}^6$ along a linear $\mathbb{P}^4$

Table: The minimal $\mathbb{P}$-dimension of toric Fano manifolds of low dimension. (Thanks to Will Reynolds)

# $m(X) = 1$ Goal is to show $X$ is not 2-Fano

Explicitly construct surf $\subset X$ with $ch_2(X).\text{surf} \leq 0$

Given $P = \{x, -x\}$ a primitive collection of $X$, and using results of Casagrande, we can construct $f : X \to Y$ birational, such that

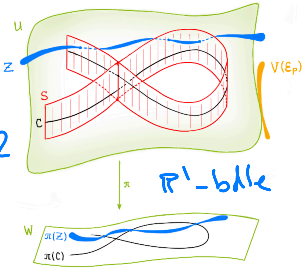
- $P_Y := \{x, -x\}$ is a primitive collection of $Y$,
- $V(\mathcal{E}_{P_Y})$ has codim $\geq 2$ in $Y$,
- $f$ is a composition of at most $2$ blow-downs with disjoint centers and smooth target

**Construct a surface $S \subset Y$:**

$U = Y \setminus V(\mathcal{E}_{P_Y})$

$Z =$ closed subset in $Y$ of codim $\geq 2$

$C \subset U \setminus Z$ very free rat'l curve




$\mathbb{P}^1$-bdle

$S = \pi^{-1}(\pi(C)) \subset U$

in fact

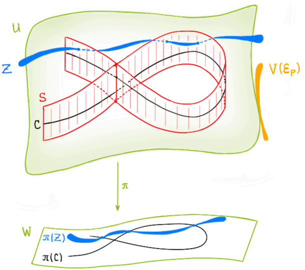
$S . ch_2(Y) = 0$

# $m(X) = 1$

**Theorem (ABCJMMV)**

Let $X$ be a smooth toric Fano variety with $m(X) = 1$. Then $X$ is not 2-Fano.

$X \xrightarrow{f}$




$\subset Y$

$S \cdot ch_2(Y) = 0$

Let $S_X$ = strict transform, then

$S_X \cdot ch_2(X) \le S \cdot ch_2(Y) = 0$

# Toric Fano manifolds $X$ with large values of $m(X)$

$m(X) = \dim X$: Projective spaces are the only toric manifolds admitting a centrally symmetric primitive relation of order $\dim(X) + 1$.

$m(X) = \dim X - 1$: Chen-Fu-Hwang classify toric Fano manifolds admitting a centrally symmetric primitive relation of order $\dim(X)$.

- There are three such varieties, and two of them also admit a centrally symmetric primitive relation of order $2$,
- The only $n$-dimensional toric Fano manifold $X$ with $m(X) = n - 1$ is the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{n-2}$.

$m(X) = \dim X - 2$: Beheshti-Wormleighton investigate toric manifolds admitting a centrally symmetric primitive relation of order $\dim(X) - 1$

- show that they have Picard number $\rho(X) \leq 5$.
- most of these varieties also admit centrally symmetric primitive relations of order $2$ or $3$

**Theorem (ABCJMMV)**

Let $X$ be a toric Fano manifold with $\dim(X) = n \geq 6$ and $m(X) \geq 3$. If $X$ has a centrally symmetric primitive relation of order $n - 1$,

$$x_0 + x_1 + \cdots + x_{n-2} = 0,$$

then $\rho(X) \leq 3$. Moreover, $m(X) = n - 2$ and the above relation is the only centrally symmetric primitive relation of $X$.

- Kleimann – gives classif of toric Fano w/ $\rho(X)=2$
- Batyrev – " descrip of sm toric w/ $\rho(X)=3$

**Theorem** (ABCJMMV): Let $X$ be a toric Fano manifold with $m(X) \geq \dim(X) - 2$

(1) The only $n$-dimensional toric Fano manifold $X$ with $m(X) = n$ is $\mathbb{P}^n$. ← 2-Fano

(2) For $n \geq 3$, the only $n$-dimensional toric Fano manifold $X$ with $m(X) = n - 1$ is the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{n-2}$. not 2-Fano

(3) For $n \geq 6$, there are eight distinct isomorphism classes of $n$-dimensional toric Fano manifolds $X$ with $m(X) = n - 2$:

(a) $X = \mathbb{P}_S(\mathcal{E})$ is a $\mathbb{P}^{n-2}$-bundle over a toric surface $S$, where $(S, \mathcal{E})$:

$\rho(X)=2$
- $S = \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$,
- $S = \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-3}$,
- $S = \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$,

not 2-Fano

$\rho(X)=3$
- $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus n-2}$,
- $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus n-3}$,
- $S = \mathbb{F}_1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{F}_1}(e+f) \oplus \mathcal{O}_{\mathbb{F}_1}^{\oplus n-2}$, where $e \subset \mathbb{F}_1$ is the $-1$-curve, and $f \subset \mathbb{F}_1$ is a fiber of $\mathbb{F}_1 \to \mathbb{P}^1$.

(b) Let $Y \simeq \mathbb{P}_{\mathbb{P}^2}\left(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}\right)$ be the blowup of $\mathbb{P}^n$ along a linear subspace $L = \mathbb{P}^{n-3}$, and denote by $E \subset Y$ the exceptional divisor. Then $X$ is the blowup of $Y$ along a codimension 2 center $Z \subset Y$, where:

$\rho(X)=3$
- $Z$ is the intersection of $E$ with the strict transform of a hyperplane of $\mathbb{P}^n$ containing the linear subspace $L$, or
- $Z$ is the intersection of the strict transforms of two hyperplanes of $\mathbb{P}^n$, one containing the linear subspace $L$, and the other one not containing it.

## Corollary

The projective space $\mathbb{P}^n$ is the only smooth $n$-dimensional toric 2-Fano manifold with $m(X) = 1, n-2, n-1, n$.

**To Do:** Address the "middle cases"

| $\dim(X)$ | # Fanos | #($m$=1) | #($m$=2) | #($m$=3) | #($m$=4) | #($m$=5) | #($m$=6) |
|---|---|---|---|---|---|---|---|
| 4 | 124 | 107 | 15 | 1 | 1 | | |
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# Thank you!


May 2022 at ICERM