The Calabi problem for Fano 3-folds

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K-stability: characterise existence of KE metrics on Fano manifolds

Theorem (Y-T-D conjecture, Chen-Donaldson-Sun, Tian).

X a Fano manifold

X admits a Kähler Einstein metric $\Leftrightarrow$ X is K-polystable

- Equivalence of deep properties in algebraic and differential geometry

Question: Which Fano manifolds are K-polystable?

dim 2: 10 deformation families of smooth dP surfaces

dim 3: 105 deformation families of smooth Fano 3-folds (with description - Iskovskikh, Mori-Mukai)

Key questions:

$\mathcal{F}$ deformation family of Fano 3-folds

1. Is the general member of $\mathcal{F}$ K-polystable?
   ↳ Known for all families
2. Which members of $\mathcal{F}$ are K-polystable?
   ↳ 71 out of 105 families
3. Is there a moduli space/stack representing the elements of $\mathcal{F}$?
   ↳ Mostly open.

K stability of Fano manifolds - main results.

- For 27 out of 34 families for which Calabi problem not entirely solved expect

Conjecture All smooth members of the 27 families
1.9, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9
2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17
2.18, 2.19, 3.2, 3.3, 3.4, 3.6, 3.7, 3.11, 4.1
are K-stable

- The remaining 7 families

1.10, 2.20, 2.21, 2.22, 3.5, 3.8, 3.12

have
- K-polystable general member
- some non K-polystable members

prime Fano 3-folds of genus 12.
Donaldson's conjecture

First obstructions: Example of del Pezzo surfaces.

- Smooth dP surface of degree d - S_d

$d=9$ $\qquad \mathbb{P}^2$

$d=8$ $\qquad \mathbb{P}^1\times\mathbb{P}^1$, $Bl_p\,\mathbb{P}^2$ — $\mathrm{Aut}\cong \mathbb{G}_a^2\rtimes PGl_2$

$d=7$ $\qquad Bl_{p,q}\,\mathbb{P}^2$ — $\mathrm{Aut}\cong (B_2\times B_2)\rtimes \mu_2$

$d=6$ $\qquad (1,1,1)\subset \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$

$d=5$ $\qquad (H_1\cap H_2\cap H_3\cap H_4)\cap Gr(2,5)$

$d=4$ $\qquad (2)\cap(2)\subseteq\mathbb{P}^4$

$d=3$ $\qquad S_3\subseteq\mathbb{P}^3$

$d=2$ $\qquad S_4\subseteq\mathbb{P}(1112)$

$d=1$ $\qquad S_6\subseteq\mathbb{P}(1123)$

- **Theorem:** [Matsushima, ABHLX]

X a K-polystable Fano manifold $\Rightarrow$ $\mathrm{Aut}\,X$ reductive

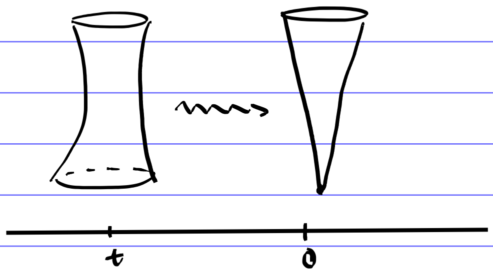
Theorem [Tian] All smooth dP surfaces with reductive automorphism group are K-polystable

Definitions of K-stability: degenerations, MMP

- A **test configuration** of $(X, -rK_X)$

abstract 1 ps flat degeneration $(\mathcal{X}, \mathcal{L})/_{\mathbb{A}^1}$

$$\begin{array}{ll} \mathcal{X} & \text{flat proper } \mathbb{G}_m \text{ equivariant} \\ \pi \downarrow & \mathcal{L} \to \mathcal{X} \quad \mathbb{G}_m\text{-equivariant } \pi\text{-ample l.b.} \\ \mathbb{A}^1 & \left(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0}\right) \cong (X, -rK_X) \times \mathbb{A}^1 \setminus \{0\} \end{array}$$



Example: Product test-configuration

- $X \circlearrowleft \mathbb{G}_m$ effective $\mathbb{G}_m$-action
- $t : \mathbb{G}_m \to T$ integral coweight

t.c. $\quad \mathcal{X} = X \times \mathbb{A}^1, \quad \mathcal{L} = -K_{X \times \mathbb{A}^1}$

$\mathbb{G}_m$-action $\quad t \cdot (x, a) \longmapsto (t(x), t \cdot a)$

$$DF(\mathcal{X}, \mathcal{L}) = 0$$

Definitions of K-stability: degenerations, MMP

- $(\mathcal{X}, \mathcal{L})$ t.c $\rightsquigarrow$ $DF(\mathcal{X})$

- X is <u>K-semistable</u> if $DF(\mathcal{X}) \geq 0$ $\forall$ non trivial t.c.

- X is <u>K stable</u> if $DF(\mathcal{X}) > 0$ $\forall$ non trivial t.c.

- X is <u>K-polystable</u> if K-semistable and $DF(\mathcal{X}) = 0 \Rightarrow (\mathcal{X}, \mathcal{L})$ product t.c

Easy first consequences:

- X K-stable $\Rightarrow$ $Aut(X)$ contains no subgroup $\simeq \mathbb{G}_m$.

- If $Aut(X) < \infty$

  K-stable = K-polystable

Definitions of K-stability: degenerations, MMP

Theorem: [Odaka] A K-semistable normal $\mathbb{Q}$ Gorenstein Fano variety is $\mathbb{Q}$-Fano

- Difficulty in definition of K-stability: ALL test configuration
- A special tc is one with $\mathcal{X}_0$ $\mathbb{Q}$-Fano

Example:

$$\{x^2+y^2+z^2+tw^2=0\} \quad \text{special}$$

$$\{x^2+y^2+tz^2+tw^2=0\} \quad \text{non special.}$$

Theorem [Li-Xu]

If $(\mathcal{X},\mathcal{L})$ is a t.c. of $(X,-rK_X)$ with $DF(\mathcal{X},\mathcal{L})\le 0$ there exists a special t.c. $(\mathcal{X}',\mathcal{L}')$ of $(X,-rK_X)$ with $DF(\mathcal{X}',\mathcal{L}')\le 0$

Definitions of K-stability: valuative approaches

X a $\mathbb{Q}$ Fano variety. [klt singularities]

- A divisor E over X (E/X)
  $\qquad \shortparallel$
  prime divisor on $Y \xrightarrow{f} X$ Y normal, f proper

- $\beta(E) = A(E) - S(E)$

$$A(E) = 1 + \mathrm{ord}_E K_{Y/X} \quad \text{log discrepancy}$$

"expected vanishing order along E"

$$S(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}\left(f^* -K_X - tE \right) dt$$

$$= \frac{1}{(-K_X)^n} \int_0^{\tau(E)} \mathrm{vol}\left(f^* -K_X - tE \right) dt$$

$$\tau(E) = \sup \{ t : f^* -K_X - tE \text{ pseudo-effective} \}$$

Theorem [Fujita - Li]

X is K-stable $\iff \beta(E) > 0 \quad \forall E/X$

X is K-semistable $\iff \beta(E) \geq 0 \quad \forall E/X$

Rem: Especially useful when symmetries.

Theorem [Datar, Liu, Székelyhidi, Zhu, Zhuang]
$\mathrm{Aut}(X)$ reductive and $\beta(E) > 0 \quad \forall E/X$ $\mathrm{Aut}(X)$-inv^t
$\Rightarrow X$ is K-polystable.

Obstructions to K-stability: divisorial unstability

X is divisorially unstable if $\exists\, S \subset X : \beta(S) < 0$

Theorem (Fujita) There are 26 deformation families of Fano 3-folds with divisorially K-unstable members

Main theorem (Calabi problem for Fano 3-folds)

Let X be a general member of a deformation family of Fano 3-folds. Then

- either X belongs to family 2.26
- or:

X K-polystable $\iff$ X divisorially K-semistable
$\iff$ X K-semistable

Example: Divisorially unstable dP surface $S = Bl_{p,q}\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$

E_1, E_2, L (-1) curves

$$-K_S \sim 3L + 2E_1 + 2E_2 \qquad \tau(L) = 3$$

- $0 \le t \le 1$: $-K_S - tL = (3-t)L + 2E_1 + 2E_2$ nef
  $\mathrm{vol}(-K_S - tL) = (-K_S - tL)^2 = 7 - 2t - t^2$

- $1 \le t \le 3$
  $$-K_S - tL \sim_{\mathbb{R}} \underbrace{(3-t)(L+E_1+E_2)}_{\text{nef part}} + \underbrace{(t-1)(E_1+E_2)}_{\text{negative part}}$$

$$\mathrm{vol}(-K_S - tL) = [(3-t)(L+E_1+E_2)]^2 = (3-t)^2$$

$$S(L) = \frac{1}{7}\int_0^1 (7-2t-t^2)\,dt + \frac{1}{7}\int_1^3 (3-t)^2\,dt$$
$$= 25/21$$

$A(L) = 1 \qquad \Rightarrow \beta(L) < 0$

Stability in families

Key question: How does stability vary in families?

Theorem(s) [Blum, Liu, Xu, Zhuang ..]

$\mathcal{X} \to Z$ a flat family of $\mathbb{Q}$ Fano varieties.

$\{ t \in Z \mid \mathcal{X}_t \text{ is K-stable} \}$ open

$\{ t \in Z \mid \mathcal{X}_t \text{ is K-semistable} \}$ open

$\{ t \in Z \mid \mathcal{X}_t \text{ is K-polystable} \}$ constructible.

Theorem [Li-Wang-Xu] (Uniqueness of K-ps degenerations)

A $\mathbb{Q}$-Fano X is K-ps

$\iff$. X is K-sstable

. any special t.c. $\mathcal{X}$ with K-ps central fibre has $\mathcal{X}_0 \simeq X$.

- What does K-polystability / semistability mean geometrically?
- Longer term goal: Moduli spaces / semistable degenerations of families with K-ps members.

The maverick – Family 2·26

$$\begin{tikzcd}
 & & Z \subseteq X \arrow[dll] \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Sing}\{p\} \subseteq C_3 \subseteq & Q & & V_5 \supset \ell
\end{tikzcd}$$

twisted cubic $\qquad\qquad\qquad\qquad$ line

- $\mathcal{N}_{\ell/V_5} = \begin{cases} ① \ \mathcal{O} \oplus \mathcal{O} \longrightarrow X_1 \\ ② \ \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow X_2 \end{cases}$

- $\exists !$ $\underset{\overset{\wr\!\!\wr}{Q}}{H}$ hyperplane section s.t. $C_3 \subseteq H$.

① H smooth

$\mathrm{Aut}^\circ(X_1) = \mathbb{G}_m$

$\exists$ degeneration $X_1 \rightsquigarrow X_0$

with $\cdot X_0$ $\mathbb{Q}$ Fano

$\cdot \mathrm{Aut}(X_0) = \mathbb{G}_m^2 \rtimes \mu_2$

X_0 : K-polystable

$\Rightarrow X_1$ strictly K-semistable

② $\mathrm{Sing}\, H = \{p\} \in C_3$

$\mathrm{Aut}^\circ(X_2) = \mathbb{G}_m \ltimes \mathbb{G}_a$

$\Rightarrow X_2$ not K-polystable

$Z = p^{-1}(\{p\}) \subseteq X$

$$\begin{array}{ccc} \tilde{X} & \supset & E \\ \downarrow & & \downarrow \\ X & \supset & Z \end{array} \qquad \beta(E) < 0$$

$\Rightarrow X_2$ not K-semistable.

Example: Cubic Threefolds. (Liu-Xu)

Let $X_3 \subseteq \mathbb{P}^4$ be a cubic 3-fold.

[Paul-Tian] If X is smooth and K-ps then X is GIT polystable

Theorem [Liu-Xu] If a possibly singular $X_3 \subseteq \mathbb{P}^4$ is GIT-polystable | GIT-semistable then it is K-polystable | K-semi stable

In particular: $U^{ss} \subseteq \mathbb{P}^{34}$

$$\downarrow$$

$$M^{GIT} = U^{ss} /\!/ PGL(5)$$

yields a proper good quotient parametrising all K-ps 3-folds smoothable to a cubic 3-fold.

Corollary: • Explicit description of singularities that can appear on K-ps degenerations

Idea of proof: ① Understand singularities that can appear on degenerations.

[Link between global volume of a K-semistable Fano and local volumes of sing. points

$$\widehat{\mathrm{vol}}(x, X) \cdot \left(\frac{n+1}{n}\right)^n \geqslant (-K_X)^n$$

• Explicit bounds for klt non smooth 3-fold sing-s
eg $\widehat{\mathrm{vol}}(x, X) \leq 16$ with $= \iff A_1$

② Use this to show that all such degenerations can be embedded in a suitable explicit ambient space, where GIT can be used.

Example : Family 2·24

$X = (1,2) \subseteq \mathbb{P}^2_{xyz} \times \mathbb{P}^2_{uvw}$

smooth members of the form

$$X_\mu = \left\{ xu^2 + yv^2 + zw^2 + \mu(xvw + yuw + zuv) = 0 \right\}$$

$$\mu \in \mathbb{C}, \quad \mu^3 \neq -1$$

$$\underbrace{Y_1}_{\rightsquigarrow X_0} = \left\{ (u^2 + vw)x + (uw + v^2)y + w^2 z = 0 \right\}$$

$$\underbrace{Y_2}_{\rightsquigarrow X_0} = \left\{ (u^2 + vw)x + v^2 y + w^2 z = 0 \right\}$$

• Have: $\mathrm{Aut}^\circ(X) = \begin{cases} \mathbb{G}_m^2 & X = X_\mu, \ \mu = 0, 2, \pm 1 \pm \sqrt{3} \\ \mathbb{G}_m & X = Y_2 \\ \{0\} & \text{otherwise} \end{cases}$

Known: $\left[\begin{array}{l} X_\mu \text{ is K-ps } \forall \mu \\ Y_1 \text{ and } Y_2 \text{ are strictly K-semistable.} \end{array} \right.$

[Moduli description ?]

Some conjectures _ Prime Fano 3-folds of genus 12 (1·10)

$\mathcal{F}$ = family 1.10 = smooth members $X = Gr(3, V, \eta)$

- $V = \mathbb{C}^7$, $N = \mathbb{C}^3$
- $\eta : \Lambda^2 V \longrightarrow N$ net of alternating forms on V

$$Gr(3, V, \eta) = \{ E \in Gr(3, V) \mid \Lambda^2 E \in \ker \eta \}$$

Facts: • $\exists !\ X_{22}^{MU} \in \mathcal{F}$ with $Aut = PGL_2(\mathbb{C})$

Donaldson : X_{22}^{MU} is K-polystable.

• $\exists !\ X_{22}^{a} \in \mathcal{F}$ with $Aut = \mathbb{G}_a \rtimes \mu_4$

$\exists\ X_{22}^{a} \rightsquigarrow X_{22}^{MU}$ is strictly K-semistable

- $\exists$ 1-dimensional family $\{X_{22}^{u}\}, u \in \mathbb{C} \setminus \{0,1\}$

and • $X_{22}^{-1/4} = X_{22}^{MU}$

- $Aut = \mathbb{G}_m \rtimes \mu_2$ for $u \neq 0, 1, -1/4$

(Cheltsov - Shramov, Fujita) : K-polystable when $u \neq 0, 1$.

- All other members of the family have finite automorphisms

[$\exists$ examples that are K-ps. C-S. / strictly K-sst. Tian]

How to describe non K-ps elements ?

Conjecture [Donaldson] A neighbourhood of $[X_{22}^{MU}]$ is identified with a local analytic neighbourhood of $0 \in T = H^1(X, T_{X_{22}^{MU}})$ which is equipped with an $Sl_2(\mathbb{C})$-action.

$\forall\ 0 \neq \alpha \in T$ small

X_α is K-polystable $\Leftrightarrow$ α is GIT polystable

2nd Formulation [everywhere]:

$X \in \mathcal{F}$ is K-polystable $\Leftrightarrow$

- X has an effective $\mathbb{G}_m$-action or
- no element of $|-K_X|$ has singularities worse than $y^4 = x^3 + t^4 x$.

Some conjectures – Prime Fano 3-folds of genus 12 (1·10)

Another look – Mukai's construction.

relate to some GIT stability question ?

birational map:

$$\mathcal{M}_3 \dashrightarrow \mathcal{M}_{22}$$

moduli of genus 3 curves
"
moduli of plane 4ic curves

moduli of prime Fano 3-folds genus 12.

$$(F=0) \subseteq \mathbb{P}^2 \rightsquigarrow X = \overline{VSP(F,6)}$$

$VSP(F,6) = \{(\ell_1, .. \ell_6) \in Hilb^6(\check{\mathbb{P}}^2) : F = \ell_1^4 + \dots + \ell_6^4\}$

$$(F=0) = \mathcal{H}_1 \leftsquigarrow X$$

Hilbert scheme of lines on X

$$\begin{array}{ccc} (F=0) & = & \mathcal{H}_1 \\ \cap & & \cap \\ \mathbb{P}^2 & = & \mathcal{H}_2 \end{array}$$

Hilbert scheme of conics on X

Special cases :

- $X = X_{22}^{Mu}$ — C = double conic
- $X = X_{22}^{a}$ — $C = C_1 \cup C_2$ 2 rational curves glued at sing. point
- $X = X_{22}^{u}$ — $C = C_1 \cup C_2$ 2 rational curves glued at 2 points single tangency

Question: K-polystability of $X \overset{?}{\Longleftrightarrow}$ GIT-stability of C

Some conjectures – Family 2.22

- 1 parameter family

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ C_4 \subseteq \mathbb{P}^3 & & V_5 \supseteq C \end{tikzcd}$$

twisted quartic $\qquad\qquad\qquad$ conic

- $\exists !$ smooth quadric surface $Q \supseteq C_4$ and

$$\mathrm{Aut}(X) = \mathrm{Aut}(Q, C_4)$$

$$Q \simeq \mathbb{P}^1_{u:v} \times \mathbb{P}^1_{x:y} = \{[xu : xv : yu : yv]\} \supseteq C_4 = (1,3)$$

$$C_4 = \{u f_3(x,y) = v g_3(x,y)\} \qquad f_3 \wedge g_3 = 1.$$

$$C_4^{a,b} = \{u(x^3 + a x^2 y) = v(y^3 + b y^2 x)\} \qquad a, b \in \mathbb{C}$$

$\boxed{\text{results}}$:

- $a = b = 0$ $\quad \mathrm{Aut} = \mathbb{G}_m \rtimes \mu_2$ (only member with $\mathrm{Aut}^0 \neq \{0\}$)

  K-polystable

- $a = 0, b \neq 0$ $\quad C_4 = \{u x^3 = v(y^3 + y^2 x)\}$

  $a \neq 0, b = 0$ $\quad$ Strictly K-semistable

- a, b general $\quad$ K stable

  (open-ness of K-stab, $\exists$ example with $\mathrm{Aut} = \mathfrak{A}_4$ and K-stable)

Reparametrise

$$C_4^{\lambda} = \{u(x^3 + \lambda x^2 y) = v(y^3 + \lambda y^2 x)\} \quad \lambda \in \mathbb{C}^\times$$

C_4 smooth $\Rightarrow \lambda \neq \pm 1$ $\qquad (\lambda = \pm 3 \leftrightarrow \lambda = 0)$

Conjecture: $\quad X_\lambda$ stable $\forall \lambda \in \mathbb{C}^\times \setminus \{\pm 1, \pm 3\}$

Some conjectures – Family 2·20

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr, "\eta"] & \\
C_3 \subseteq V_5 \arrow[rr, dashed] & & \mathbb{P}^2
\end{tikzcd}$$

twisted cubic

$$\mathbb{P}^6 \dashrightarrow \mathbb{P}^2$$

- $\exists$ $Sl_2(\mathbb{C})$ - equivariant iso

universal twisted cubic

$$\begin{tikzcd}
 & \mathcal{Y} \arrow[dl, "\phi"'] \arrow[dr, "\sigma"] & \\
V_5 & & \mathbb{P}^4 = \mathcal{H} = \mathbb{P}(V)
\end{tikzcd}$$

$\sigma = Bl_{\mathcal{S}}\, \mathbb{P}^4$ $\quad$ $\mathcal{S} = \nu_2(\mathcal{H}_1)$ $\quad$ $Sl_2(\mathbb{C})$ - inv^t surface degree 4 in $\mathbb{P}^4$

$L \subseteq \mathbb{P}^4$ $\qquad$ $C_L = \phi_*(\sigma^* L)$ twisted cubic

$L \cap \mathcal{S} = \phi \iff C_L$ smooth

$X_L = Bl_{C_L} V_5$

Conjecture: X_L K-polystable $\iff$ orbit of line as a point in $Gr(2, V)$ is GIT-p.s. ($Sl_2 \mathbb{C}$ - action)

Some conjectures _ Family 2·21

$$X \longrightarrow Q \subseteq \mathbb{P}^4$$

- blowup along a twisted quartic

Fix a standard $Sl_2\,\mathbb{C}$ action on $\mathbb{C}^2$

$\Rightarrow V = Sym^4 W$

and $\mathbb{P}(V) = \mathbb{P}^4$

$[u:v] \mapsto [u^4 : u^3v : u^2v^2 : uv^3 : v^4]$

Z = $Sl_2(\mathbb{C})$ invariant twisted quartic

$$\begin{array}{ccc} Z & & \\ \cap\!| & & \\ Q & \xleftarrow{\pi} & X \end{array}$$

quadric singular or not

[description of GIT stable

Conjecture If Q (and hence X) is smooth

X is GIT polystable

$\Longleftrightarrow$ X is K-polystable