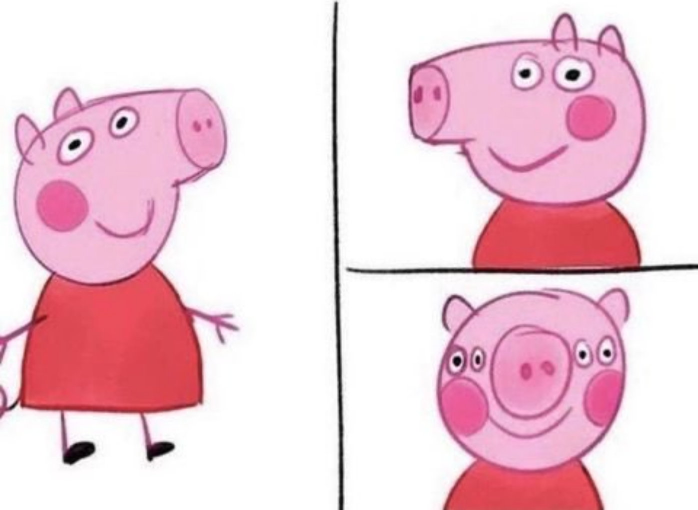
Have you ever wondered what Peppa Pig looks like from the front?



Notthingham AG Seminar May 19 / 2022

Buildings as Classifying spaces for toric principal bundles

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joint with Chris Manon (UKY) &
Boris Tsvelikhovsky (Pitt)

- Review of toric varieties

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ multi. group $\qquad T = (\mathbb{C}^*)^n$ n-dim. alg. torus

$N = \{\gamma : \mathbb{C}^* \longrightarrow T\} \cong \mathbb{Z}^n \qquad t \longmapsto (t^{\gamma_1}, \dots, t^{\gamma_n})$

$M = \{\chi : T \longrightarrow \mathbb{C}^*\} \cong \mathbb{Z}^n \qquad \underbrace{(x_1, \dots, x_n)}_{x} \longmapsto \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}_{x^\alpha}$

Toric variety

$T \curvearrowright X$

normal $\begin{matrix}\text{irr. var.} \\ \dim = n\end{matrix}$

$T \cong \underbrace{U_0}_{T\text{-orbit}} \overset{\text{open}}{\subset} X$

. Generalize affine & proj. space

$N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^n$

$\sigma \subset N_{\mathbb{R}}$ strictly convex rational polyhedral cone

$\Sigma = \{\sigma \subset N_{\mathbb{R}}\}$ → $\tau \leq \sigma \Rightarrow \tau \in \Sigma$; → $\sigma_1 \cap \sigma_2$ face of both

fan

Σ

<u>Thm</u> $\Sigma \xleftrightarrow{1-1} X_\Sigma$

equiv. of categories between <u>Cat. of fans</u> & <u>Cat. of toric varieties.</u>

- $\sigma \in \Sigma \rightsquigarrow U_\sigma$ affine toric var. $\subset X_\Sigma$
- Fix $x_0 \in U_0 \rightsquigarrow T \cong U_0 \qquad t \longmapsto t \cdot x_0$

<u>Line bundles</u>

$$\begin{array}{c} \mathcal{L} \circlearrowleft T \\ \downarrow \\ X_\Sigma \circlearrowleft T \end{array}$$

equiv. line bundle

+ id. of $\mathcal{L}_{x_0} \cong \mathbb{C}$

<u>Thm</u> T-line bundles on X_Σ $\xleftrightarrow{1-1}$ T-inv. Cartier div. on X_Σ

$\mathcal{L}$

union of Cones in Σ

$\xleftrightarrow{1-1} \varphi : |\Sigma| \longrightarrow \mathbb{R}$ int. piecewise linear

① $\varphi : N \cap |\Sigma| \longrightarrow \mathbb{Z}$

② $\forall \sigma \in \Sigma \quad \varphi|_\sigma$ linear.

Σ

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \implies \varphi(v_\rho) = a_\rho$$

$\rho \in \Sigma(1)$: ray

v_ρ: primitive vec. along ρ

$D|_{U_\sigma}$ principal div. $\implies$ $\varphi|_\sigma$ linear

- toric line bundle: Data of $\{a_\rho\}_{\rho \in \Sigma(1)}$ $\forall \sigma \in \Sigma$ $\rho \in \sigma(1)$ a_ρ are "compatible"

Toric vector bundles

$$\begin{matrix} \mathcal{E} \circlearrowleft T \\ \downarrow \\ X_\Sigma \circlearrowleft T \end{matrix}$$

Example: TX_Σ tangent bundle of a toric variety

Thm (Klyachko ~ 1989) $\rightsquigarrow$ Kaneyama ~ 1970's

$\mathcal{E}$ toric vec. bundle on X_Σ

rank $\mathcal{E} = r$ $\quad E = \mathcal{E}_{x_0} \cong \mathbb{C}^r$

$$\mathcal{E} \xleftrightarrow{1-1} \text{"compatible" families of decreasing } \mathbb{Z}\text{-filtrations } \{E^\rho_\bullet\}_{\rho \in \Sigma(1)}$$

$E \supset \cdots \supset E^\rho_0 \supset E^\rho_1 \supset \cdots \supset 0$ in E

$\mathbb{C} \supset \cdots \underbrace{\mathbb{C} \supset 0} \cdots \supset 0$ $\quad E = \mathbb{C}$ $\quad r = 1$

Klyachko's Compatibility Condition:

- $\sigma \in \Sigma$ $\quad \rho_1, \dots, \rho_s$ rays in σ

$\exists\ B_\sigma = \{b_1, \dots, b_r\}$ basis for E

$\exists\ u_\sigma = \{u_1, \dots, u_r\} \subset M$ char. of T

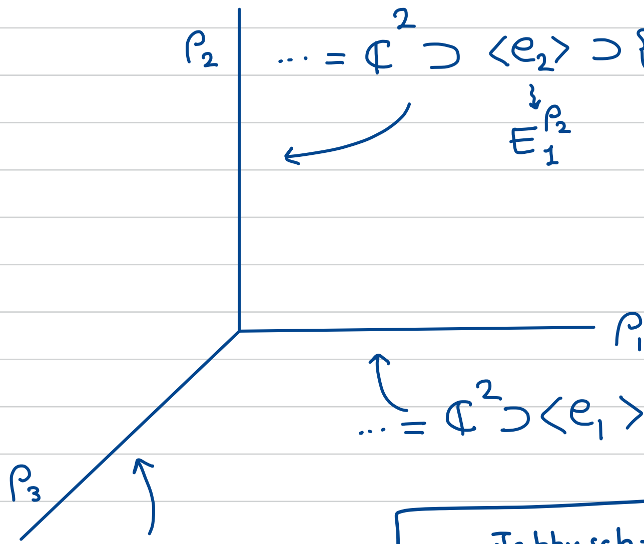
$$E_i^\rho = \operatorname{Span}_{\mathbb{C}} \{ b_i \mid \langle u_i , v_\rho \rangle \geq i \}$$

smallest int. vec. along ray ρ

- Key Obs.: $\mathcal{E}|_{U_\sigma} \cong U_\sigma \times E$ is T-equiv. trivial $\rightsquigarrow$ everywhere ind. T-weight sections

$T \longrightarrow GL(E)$

<u>Example</u> $\quad X_\Sigma = \mathbb{CP}^2 \quad \mathcal{E} = TX_\Sigma \quad E = \mathbb{C}^2$



$\dots = \mathbb{C}^2 \supset \langle e_2 \rangle \supset \{0\} = \dots$

$E_1^{\rho_2}$

$\dots = \mathbb{C}^2 \supset \langle e_1 \rangle \supset \{0\} = \dots$

$\dots = \mathbb{C}^2 \supset \langle -e_1 - e_2 \rangle \supset \{0\} = \dots$

- Jabbusch - Di Rocco - Smith

Parliament of polytopes

Tits building (of a linear alg. gp. G)

Building : Certain (infinite) abstract simplicial complex + distinguished (finite) sub-complexes (called apartments) that satisfy certain axioms.

- Used in classification of s.s. alg. gps over arbitrary fields
- Discrete analogue of symm. spaces of Lie gps

L. Ji " Buildings & their app. in geo. & top. "

- $G \curvearrowright G$ by conj. $\rightsquigarrow$ $G \curvearrowright$ building

sends apt. to apt. transitively

Tits building / Spherical buildings

G/P proj. var.

G linear alg. gp. $\Delta(G)$

Simplexes $\xleftrightarrow[\text{inclusion rev.}]{1-1}$ P parabolic subgroups

$\Delta_Q \subset \Delta_P$ $P \subset Q$

Max. Simplexes (Chambers) $\xleftrightarrow{1-1}$ Borel subgroups

apartments $\xleftrightarrow{1-1}$ max. tori

- $\underset{\text{max. torus}}{H \subset G}$ $\rightsquigarrow$ apt. of H = Coxeter Complex of (G, H)

$\Lambda^\vee(H)$ = Cochar. lattice = Weyl chambers & their faces in $\Lambda^\vee_{\mathbb{R}}(H)$

$\Delta(G)$ example of spherical building :

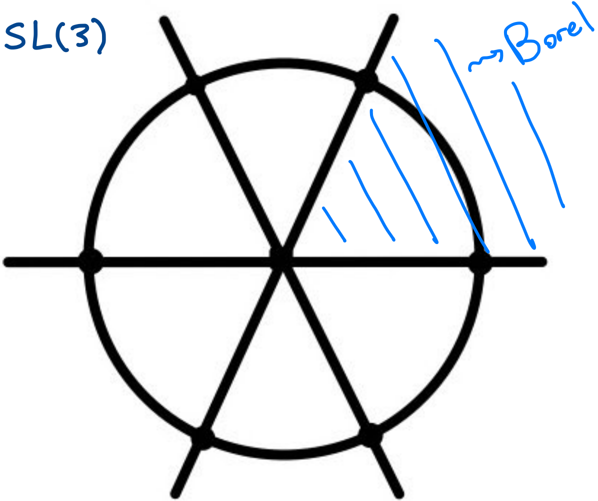
each apt. a triangulation of a sphere

Def: (Geo. realization of Tits building):

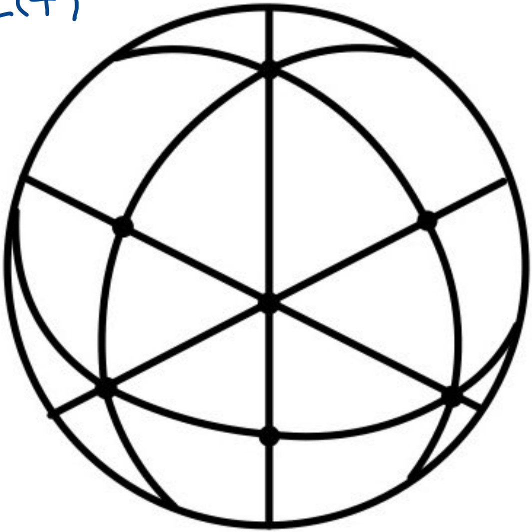
$B(G)$ = (infinite) union of spheres ($\longleftrightarrow$ max. tori) glued along simplexes corr. to the same para. subgps.

$\tilde{B}(G)$ = Cone over Tits building
= (infinite) union of r-dim. vec. spaces $\Lambda^\vee_{\mathbb{R}}(H)$ glued ....

SL(3)



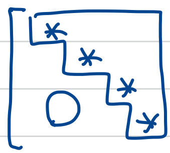
SL(4)



Tits building of $GL(r)$ $\qquad E = \mathbb{C}^r$

Simplexes $\longleftrightarrow$ $0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = \mathbb{C}^r$

P = stab. of flag



vertices $\longleftrightarrow$ Subspaces in $\mathbb{C}^r$

max simp. $\longleftrightarrow$ Complete flags

apts $\longleftrightarrow$ bases for $\mathbb{C}^r$ (up to scaling)

$F_\bullet = (F_1 \subsetneq \cdots \subsetneq F_k)$ $\qquad \{b_1, \ldots, b_r\} \subset \mathbb{C}^r$

simplex $\in$ apt.

if each F_i is spanned by subset of $\{b_1, \ldots, b_r\}$

flag is <u>adapted</u> to the basis

Tits buildings and 1-para. subgroups

$\gamma : \mathbb{C}^* \longrightarrow G$ 1-para. subgroup

Def. $\gamma_1 \sim \gamma_2$ if $\lim_{t \to 0} \gamma_1(t)\gamma_2^{-1}(t)$ exists in G.

- $G = GL(E)$

  $\gamma : \mathbb{C}^* \longrightarrow GL(E)$ $\rightsquigarrow$ weight spaces & weights $c_1 > \dots > c_k$

  $\mathbb{C}^* \curvearrowright E = \mathbb{C}^r$

  $\Downarrow$

  Flag

  $$\{0\} \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = E$$

  F_i = span of all weight vec. weight $\leq c_i$.

- **Prop.** $\gamma_1 \sim \gamma_2 \iff \gamma_i$ have same weights & same flags.

$\gamma \longmapsto P_\gamma$ = para- subgroup. ass to the flag of γ

$\gamma \longmapsto P_\gamma$ gives a realization of Tits building in terms 1-para. subgps.

<u>Prop.</u> $\rightsquigarrow$ Mumford (GIT book), K.-Manon

$$\{\text{all 1-para. subgps}\}/\sim \;\longleftrightarrow\; \text{Lattice pts in } \widetilde{B}(G)$$

Back to toric varieties:

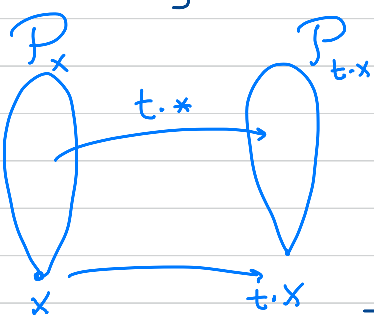
G reductive alg. gp. $/\mathbb{C}$ $\rightsquigarrow$ any alg. closed field works

X_Σ T-toric variety

<u>Def.</u> $\mathcal{P}$ <u>toric</u> principal G-bundle on X_Σ

$\mathcal{P}$ has a T-action commuting with G-action.

$$\begin{tikzcd} T \circlearrowright \mathcal{P} \circlearrowleft G \arrow[d] \\ T \circlearrowright X_\Sigma \end{tikzcd}$$



$T \longrightarrow G$

- <u>Biswas-Dey-Poddar:</u> $\mathcal{P} \downarrow U_\sigma$ is T-equiv. trivial. $\cong U_\sigma \times G$

infinite union of vec. spaces

Def. $\Phi : |\Sigma| \longrightarrow \tilde{\mathcal{B}}(G)$ piecewise linear map

① $\forall \sigma \in \Sigma \quad \exists H_\sigma \subset G$ max. torus

$\Phi|_\sigma : \sigma \longrightarrow \check{\Lambda}_{\mathbb{R}}(H_\sigma)$ linear.

$\check{\Lambda}(T) \cong \mathbb{Z}^n$

② $\Phi|_\sigma : \sigma \cap N \longrightarrow \check{\Lambda}(H_\sigma)$.

Thm (K. – Manon $\sim$ 2018 2022)

(iso. classes)

toric principal G-bundles $\mathcal{P}$ $\xleftrightarrow{1-1}$ PL maps $\Phi : |\Sigma| \to \tilde{\mathcal{B}}(G)$

Extends to equiv. of categories.

- Examples : Classification of Symp. or orth. bundles on toric varieties in terms of isotropic flags ...

- Question (Leonid Monin) Can we realize $\tilde{\mathcal{B}}(G)$ as a "tropicalization" of classifying space BG?

Bruhat – Tits buildings / affine buildings

K discretely valued field

$\mathrm{val}: K\setminus\{0\} \longrightarrow \mathbb{Z}$ valuation

e.g. (1) $K = \mathbb{C}((t))$ $\quad$ val = order of t

$\mathrm{val}(t^a(\text{const.} + \cdots)) = a$

(2) $K = \mathbb{Q}_p$ $\quad$ val = p-adic valuation

$\mathcal{O} = \{x \in K \mid \mathrm{val}(x) \geq 0\}$ DVR

$\mathfrak{m} = \{x \in K \mid \mathrm{val}(x) > 0\}$ max. ideal

" $\langle \pi \rangle$ $\quad$ π uniformizer

Spec$(\mathcal{O}) \rightsquigarrow$ two points

$$\begin{array}{lcc} & o & \& \ \eta \\ & \updownarrow & \updownarrow \\ \text{ideals} & \mathfrak{m} & \{0\} \end{array}$$

Spec$(\mathcal{O})$ = infinitesimal nghbd of $\underline{0}$

$\mathbb{C}[t] \hookrightarrow \mathbb{C}[[t]] \rightsquigarrow \mathbb{A}^1 \longleftarrow \mathrm{Spec}(\mathcal{O})$

- G reductive alg. gp. defined over $\mathcal{O}$
- To G one corresponds another building called Bruhat-Tits building of G

Motivation: classify red. gps over local fields.

$H \subset G$ (split) max. torus $\rightsquigarrow$ affine Coxeter complex $=$ triangulation of affine space $\mathbb{R}^r$

$\dim H = r$

- Triangulation $\leftrightsquigarrow$ Fundamental domains for $W \ltimes \mathbb{Z}^r$

$\mathbb{Z}^r$ → translations

B-T building of $GL(E)$:

- Vertices: All $\mathcal{O}$-lattices in $E = K^r$ (mod scalar multi)

Lattice $=$ full rank $\mathcal{O}$-module $\cong \mathcal{O}^r \subset K^r$.

(affine Grassmannian of $GL(r)$ $= GL(r,K) / GL(r,\mathcal{O})$)

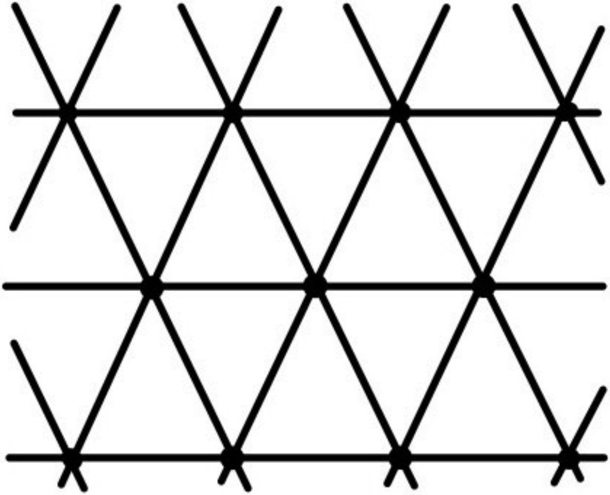
- Apartments: Fix a basis $B = \{b_1, \dots, b_r\}$

Vertices in

$$A_B = \left\{ \sum_{i=1}^{r} t^{a_i} \mathcal{O} b_i \;\middle|\; a_1, \dots, a_r \in \mathbb{Z} \right\} \cong \mathbb{Z}^r$$

SL(2)

SL(3)



$B_{aff.}(GL(r))$ = (infinite) union of affine spaces ($\longleftrightarrow$ bases in $E = K^n$) glued along common simplexes.

Toric vec. bundles over toric schemes over DVR $\mathcal{O}$

Kempf - Mumford et. al.

Last Chap. Torodial embeddings I

Good ref. Burgos Gil, Phillipon, Sombra

Thm (Mumford et. al. ~70)

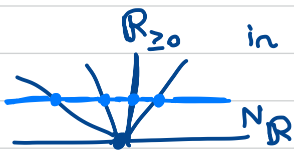
Complete toric schemes over Spec$(\mathcal{O})$ $\longleftrightarrow$ Complete rat. poly. Complexes in $N_{\mathbb{R}}$

$(N_{\mathbb{R}} \leftrightsquigarrow N_{\mathbb{R}} \times \{1\})$

$\Pi \subset N_{\mathbb{R}} \times \{1\}$

poly. complex $\rightsquigarrow$ finite collection of polyhedra in $N_{\mathbb{R}} \cong \mathbb{R}^n$

$\tilde{\Sigma} \subset N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$



$\Pi = \tilde{\Sigma} \cap (N_{\mathbb{R}} \times \{1\})$

$\mathcal{X}_{\Pi} = \mathcal{X}_{\tilde{\Sigma}} = X_{\tilde{\Sigma}} \times_{\mathbb{A}^1} \mathrm{Spec}(\mathcal{O})$

Thm. (K.–Manon–Tsvelikhovsky, 2022) $\rightsquigarrow$ not yet on arXiv

$$\left\{\begin{array}{c}\text{(iso. classes)}\\ \text{Toric vec. bundles on } \mathfrak{X}_{\Pi}\\ \text{rank } r\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{Piecewise affine maps}\\ \Phi : |\Pi| \longrightarrow \widetilde{\mathfrak{B}}_{\mathrm{aff}}(GL(r))\end{array}\right\}$$