Rolling-shutter cameras & Kummer's classification of order-one line congruences

Kathlén Kohn

joint work with Marvin Hahn, Orlando Marigliano, Tomas Pajdla

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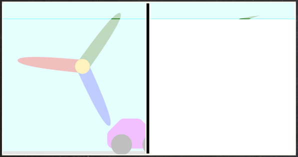

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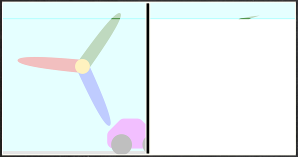

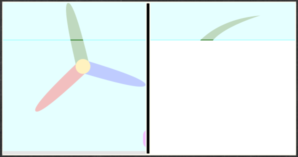

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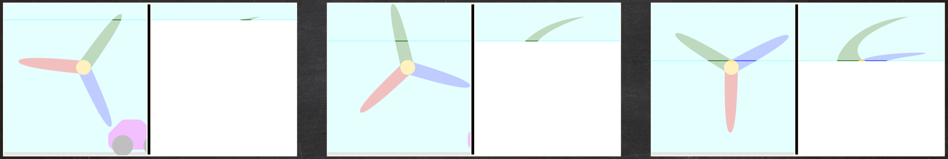

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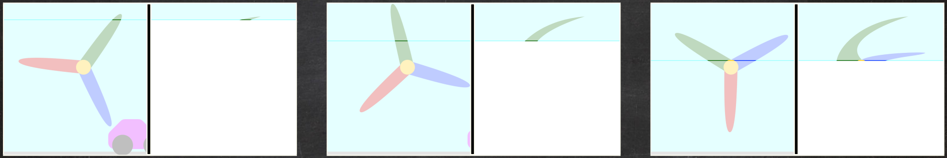

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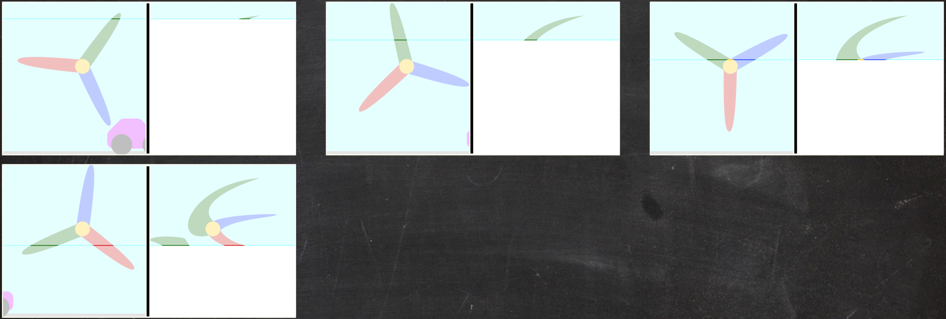

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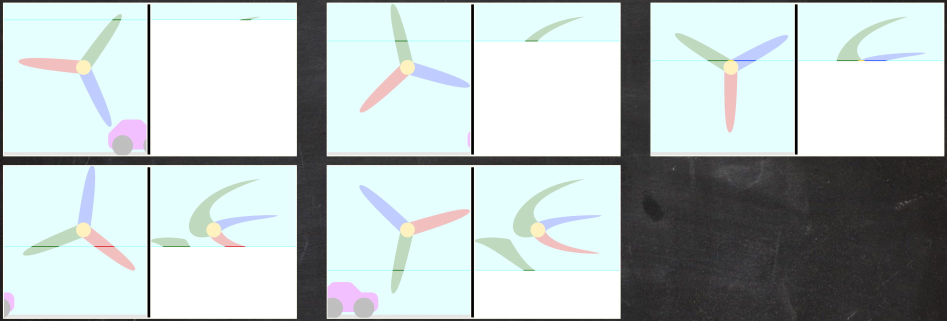

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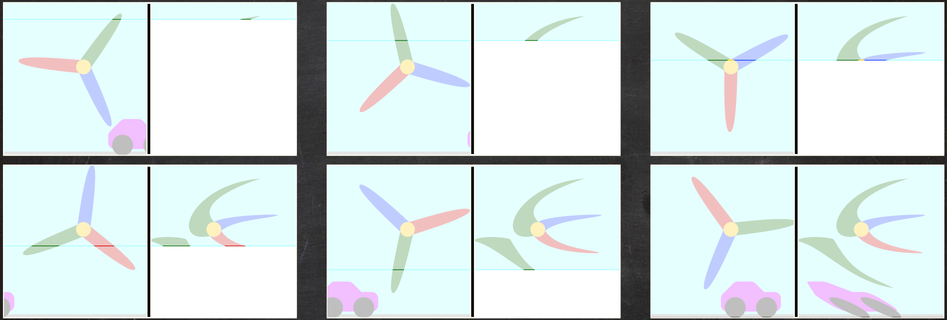

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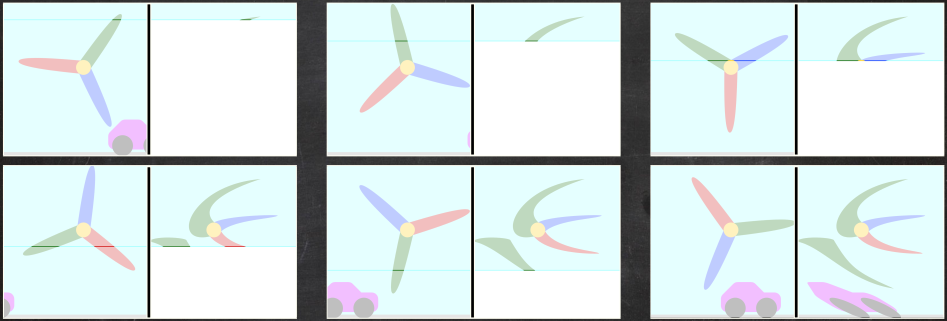

(by Cmglee @ Wikipedia

changes: added black separating line)

Algebraically:

- The image of a line is typically a higher-degree curve.
- A 3D point can appear more than once in the image.

Long-term goal:

Reconstruct 3D scenes from 2D pictures taken by *unknown* rolling-shutter cameras.

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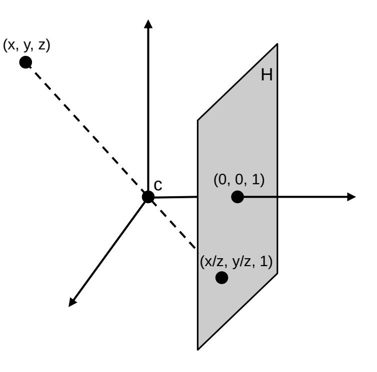
First, need to understand

- how to model rolling-shutter cameras algebraically
- how they did take pictures

Global-Shutter Camera



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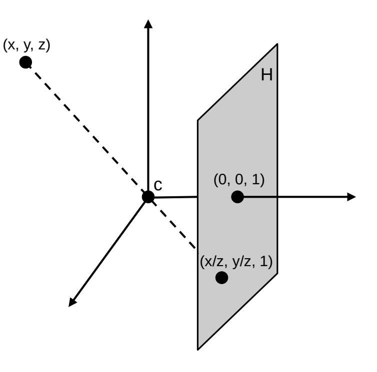

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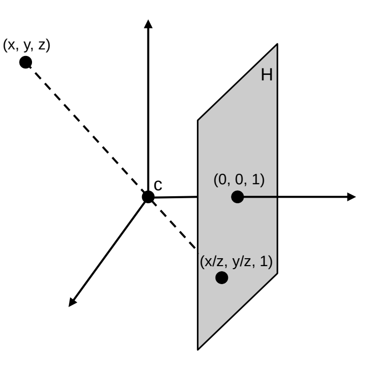

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Every calibrated global-shutter camera is obtained by translation and rotation from the standard camera, i.e., is of the form $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2, \; X \mapsto AX$, where $A = R \cdot [I_3 \mid -c] \in \mathbb{R}^{3\times 4}, R \in \mathrm{SO}(3), c \in \mathbb{R}^3$.

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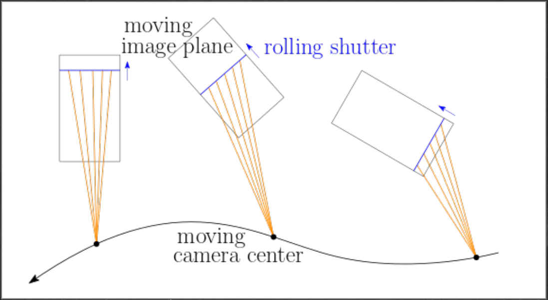

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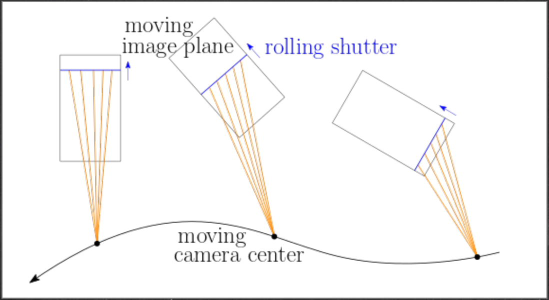

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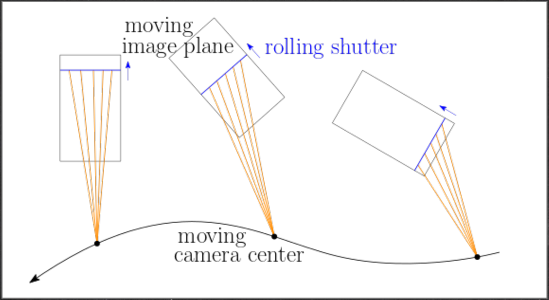

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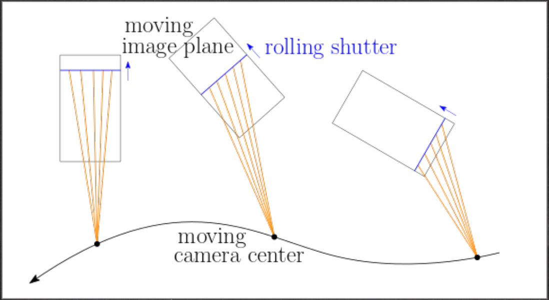

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Assume: rolling shutter parallel to y-axis on image plane:

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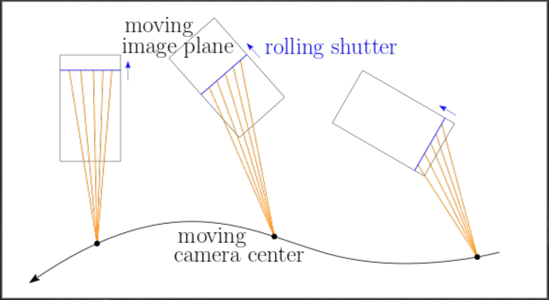

On the affine chart $\{(v:t) \mid t \neq 0\} \subset \mathbb{P}^1$, the camera's position and orientation at time $\frac{v}{t}$ are

$$c(\tfrac{v}{t}) \in \mathbb{R}^3 \quad \text{and} \quad R(\tfrac{v}{t}) \in \mathrm{SO}(3).$$

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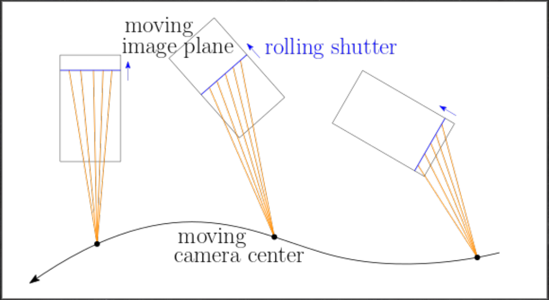

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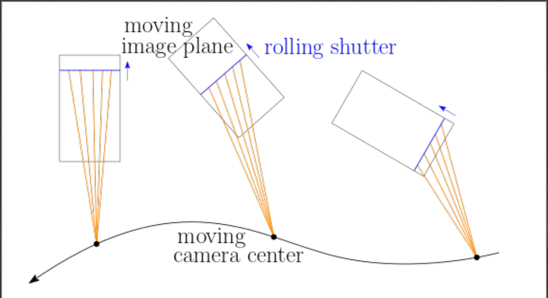
Assume: c is a rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^3$.

How to take a picture?



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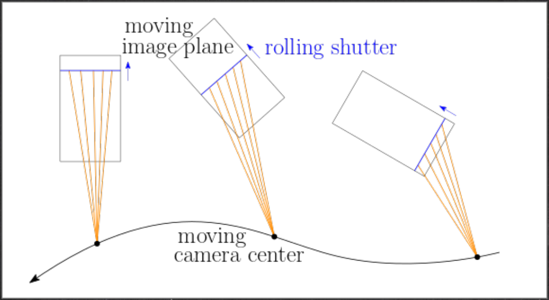

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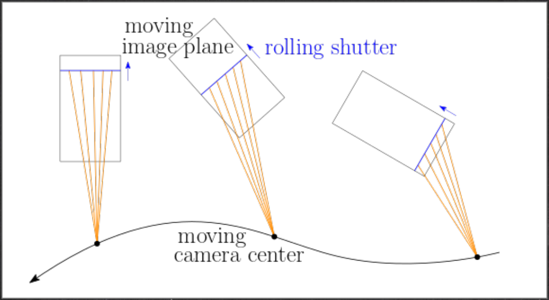

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Hence, the rolling plane is the preimage of the rolling shutter under A:

$$\sigma(\tfrac{v}{t}) := (-t : 0 : v) \cdot A(\tfrac{v}{t}) \in (\mathbb{P}^3)^*.$$

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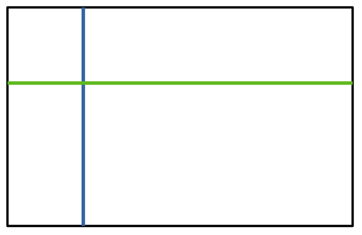

Image points are intersections of the rolling shutter with lines parallel to the x-axis:

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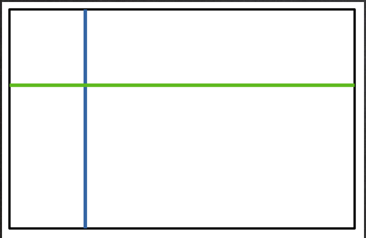

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We think of the image plane as $\mathbb{P}^1 \times \mathbb{P}^1$ via the birational map

$$\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2,$$
$$((v : t), (u : s)) \mapsto (sv : ut : st).$$

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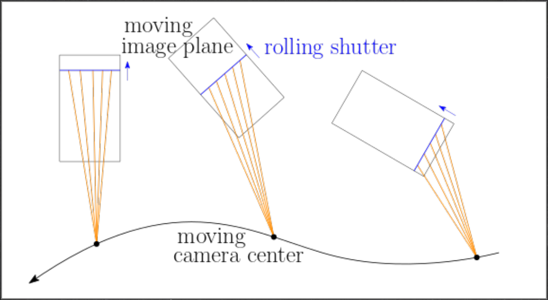

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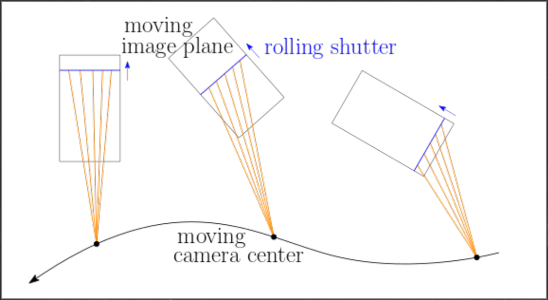

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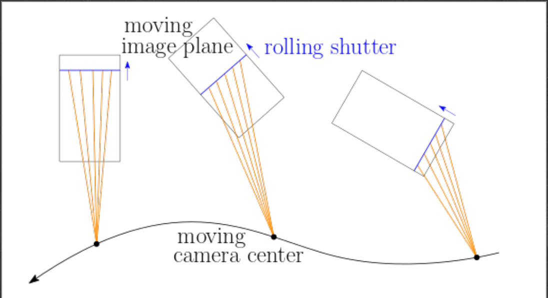

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Definition: The order of a line congruence $\mathcal{L} \subset \mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})$ is the number of lines on $\mathcal{L}$ that pass through a generic point in $\mathbb{P}^3_{\mathbb{C}}$.

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Observation: The number of times a generic point in $\mathbb{P}^3_{\mathbb{C}}$ is seen by a rolling-shutter camera is

$$\mathrm{order}(\overline{\mathrm{im}(\Lambda)}) \cdot \deg(\Lambda).$$

We call this the order of the camera.

Order-One Cameras

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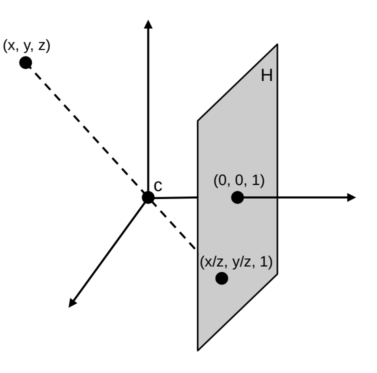
Observation: The picture-taking map is $\Lambda^{-1} \circ \Gamma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Example: Global-Shutter Camera



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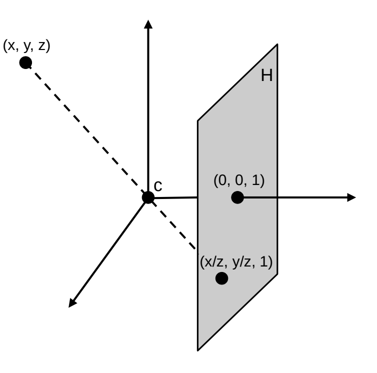

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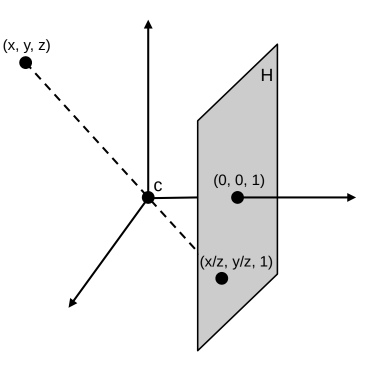

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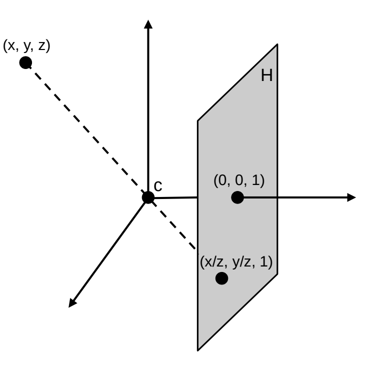

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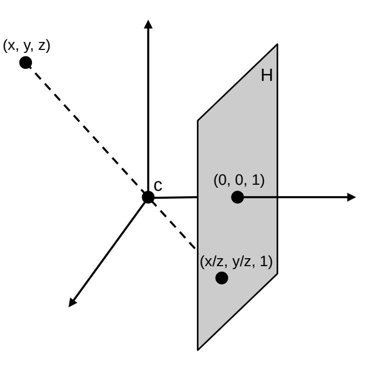

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- Λ^{-1} intersects lines on $\mathcal{L}$ with image plane H

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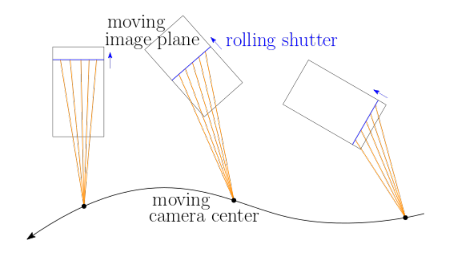

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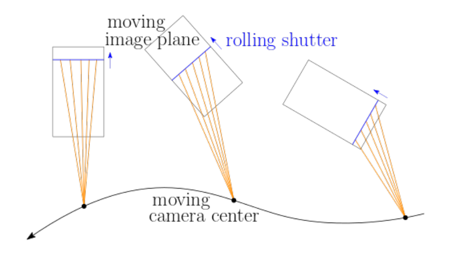

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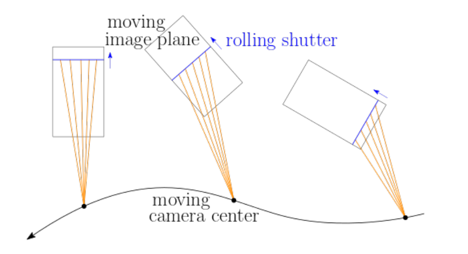

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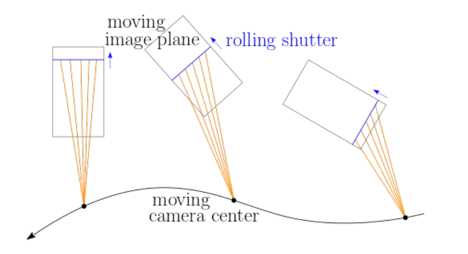

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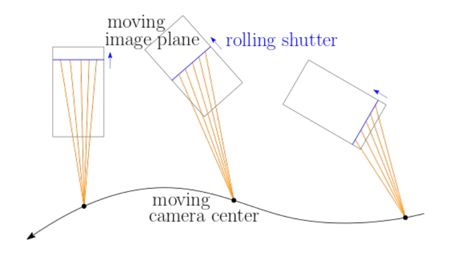

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I. C is a curve with $\#(K \cap C) = \deg(C) - 1$ (counted with multiplicities).

II. $C = K$.

III. C is a point on K.

Order-One Congruences

Theorem [Kummer, 1866]:
A congruence $\mathcal{L} \subset \mathrm{Gr}(1, \mathbb{P}^3)$ has order one if and only if it is one of the following:

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II. There is a line $K \subset \mathbb{P}^3$ and a dominant morphism $\kappa : K^\vee \to K$ such that $\mathcal{L} = \bigcup_{\Sigma \in K^\vee} \{L \in \mathrm{Gr}(1, \mathbb{P}^3) \mid \kappa(\Sigma) \in L \subset \Sigma\}$.

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Order-One Congruences

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The secant congruence of the twisted cubic curve cannot be parametrized by a rolling-shutter camera!

Moduli Spaces of Order-One Cameras of Type I

A rolling-shutter camera is defined via its center map $c : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ and its (possibly non-rational) rotation map $R : \mathbb{A}^1 \to \mathrm{SO}(3)$.

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i.e., such that

a) the intersection of all rolling planes is a line K,
b) the rolling-planes map $\sigma : \mathbb{P}^1 \dashrightarrow K^\vee$ is birational, and
c) the center locus $C := \overline{\mathrm{im}(c)}$ is a curve with $\#(K \cap C) = \deg(C) - 1$

?

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$$\mathcal{H}_d := \left\{ (C, K) \,\middle|\, \begin{array}{l} C \subset \mathbb{P}^3 \text{ rational curve, } \deg C = d, \\ K \in \mathrm{Gr}(1, \mathbb{P}^3),\ \#(K \cap C) = d - 1 \end{array} \right\}$$

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Fact [e.g. Ellia, Franco 2001]

- $\dim \mathcal{H}_d = 3d + 5$
- For every line, conic, or nondegenerate rational curve C of degree $d \leq 5$, there is a line K such that $(C, K) \in \mathcal{H}_d$.
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Is every $(C, K) \in \mathcal{H}_d$ coming from a rolling-shutter camera?
Almost: Neither C nor K are allowed to be contained in the plane at infinity $H^\infty := (0:0:0:1)^\vee$.

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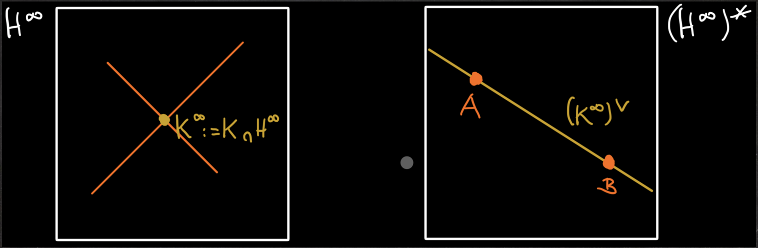

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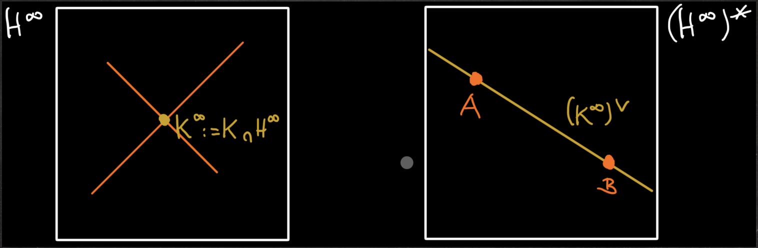

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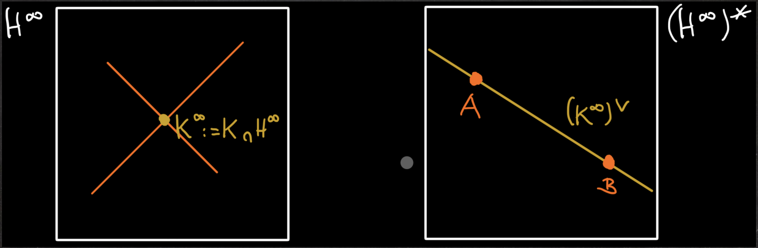

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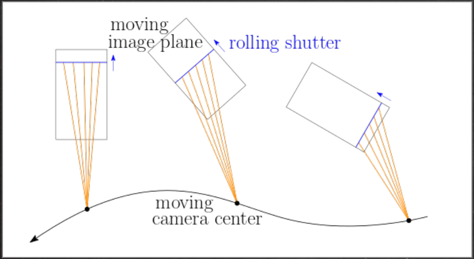

The rotation $R(v:t)$ is not yet determined! We can still choose the rotation in the rolling plane $\sigma(v:t)$ around the center $c(v:t)$.

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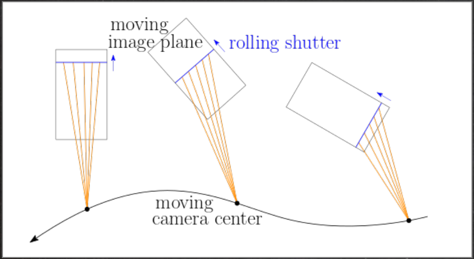

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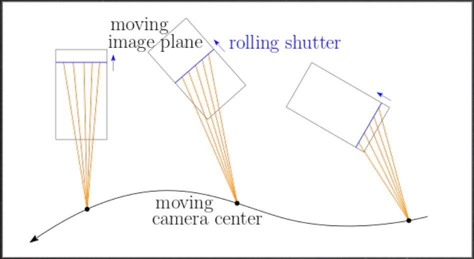

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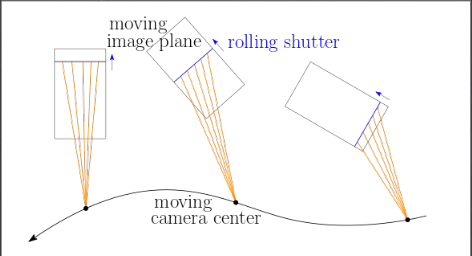

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This only determines the rotation $R(v:t)$ up to rotations by 180° about either $L(v:t)$ or the normal of $\sigma(v:t)$ through $c(v:t)$.

Moduli Spaces of Order-One Cameras of Type I

Summary:
There is a 4-to-1 correspondence between order-one rolling-shutter cameras of type I and the elements in

$$\mathcal{R}_{I,d,\delta} := \left\{ (C, K, \sigma^\infty, \lambda) \;\middle|\; \begin{array}{l} (C,K) \in \mathcal{H}_d,\ C \not\subset H^\infty,\ K^\infty = K \cap H^\infty \text{ point}, \\ \sigma^\infty : \mathbb{P}^1 \dashrightarrow (K^\infty)^\vee, (v:t) \mapsto Av + Bt, \\ \text{where } \sum_i A_i B_i = 0 \text{ and } \sum_i A_i^2 = \sum_i B_i^2, \\ \lambda : \mathbb{P}^1 \dashrightarrow K,\ \deg(\lambda) = \delta \end{array} \right\}$$

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$$\dim \mathcal{R}_{I,d,\delta} = (3d+5) + 1 + (2\delta+1) = 3d + 2\delta + 7$$

Images of Lines

Recall: The picture-taking map is

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Example: $d = 1$ and $\delta = 0$:
rolling-shutter camera of order one maps lines to conics through a fixed point