Counting Curves in $\mathbb{P}^r$

Thm A. $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$ general

$\mathbb{P}^r \ni x_1, \dots, x_n$ general pts

The number of degree d maps $f: C \longrightarrow \mathbb{P}^r$ that satisfy $f(p_i) = x_i$ is

$$\int_{Gr(r+1, d+1)} \sigma_{1^r}^{g} \cdot \Big(\underline{\sum_{\lambda \subset (n-r-2)^r}} \sigma_\lambda \sigma_{\bar{\lambda}} \Big)_{\lambda_0 \leq n-r-1.}$$

Assume $\boxed{n = \frac{r+1}{r} d - g + 1.}$

Thm B $(C, p_1, \dots, p_n) \in M_{g,n}$ general

$\mathbb{P}^2 \ni x_1, \dots, x_{n_0}$ general pts

$\mathbb{P}^2 \supset L_{n_0+1}, \dots, L_n$ general lines.

Then, the number of $f : C \to \mathbb{P}^2$ deg d with

$f(p_i) = x_i \quad i \le n_0$

$f(p_i) \in L_i \quad i > n_0$ is:

$$\int_{Gr(3,d+1)} \sigma_{1^2}^{g} \cdot \sum_{|\lambda| = n+n_0-8} \sigma_\lambda$$

number of SSYT of shape λ avoiding certain patterns.

Reformulation

$$\bar{\tau} : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^r)^n$$

Thm A asks for the degree of τ, assuming the appropriate transversality. [Brill-Noether Thm $\Rightarrow$ τ is generically étale when expected relative dim $= 0$]

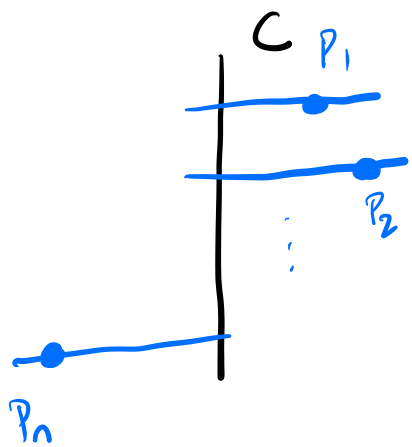
Could ask instead for $\mathrm{vdeg}(\bar{\tau})$.

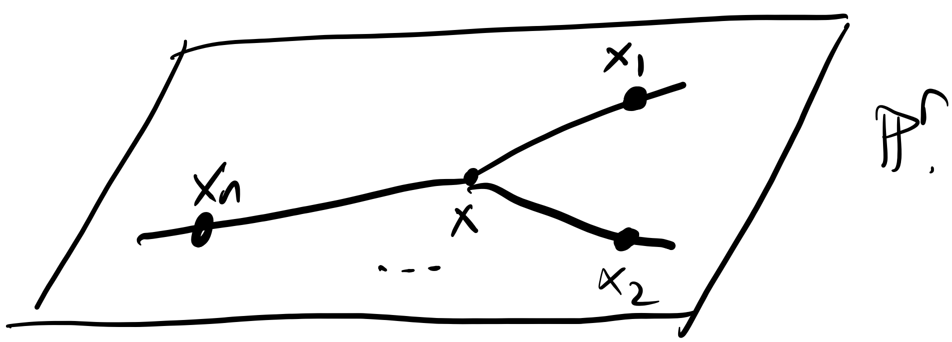
Problem: $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ often contains [dominating] components of dimension too large.

$$\bar{\tau}_* [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{vir} = \boxed{\mathrm{vdeg}(\bar{\tau})} [\overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^r)^n].$$

In general, $\mathrm{vdeg}(\bar{\tau}) \neq \deg(\tau)$.

e.g.: assume $d \geq n$.





$\in$ general fiber of $\bar{\tau}: \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^r)^n$

but not in $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$.

This contributes to $\mathrm{vdeg}(\bar{\tau})$, but not $\deg(\tau)$.

History

① 19th century: Castelnuovo.

$\deg(\tau)$ when $\boxed{n=r+2}$ (n, d as small as possible.)

$$\left\{\begin{array}{l}\text{Maps } C \to \mathbb{P}^r \\ \text{w/ } r+2 \text{ incidence} \\ \text{Conditions}\end{array}\right\} \longleftrightarrow \left\{\text{Maps } C \to \mathbb{P}^r\right\} \Big/ PGL(r+1)$$

"linear series of minimal degree".

$$\deg(\tau) = \int_{Gr(r+1, d+1)} \sigma_{1^r}^{g} = \text{generalized Catalan \#}.$$

② from 1990s.

Thm.

$$\mathrm{vdeg}(\bar{\mathcal{I}}) = (r+1)^g.$$

Bertram-Daskalopoulos-Wentworth (96).

Siebert-Tian (97)

Marian-Oprea (07)

(+Pandharipande)

Vafa-Intriligator formula: determines all virtual counts of maps $f: C \to Gr$ with arbitrary Schubert incidence conditions.

Buch-Pandharipande (21)

reduce calculation to calculation in $QH^*(\mathbb{P}^r)$.

③ since 2020.

Thm (Tevelev)

$r=1$, $d=g+1$, $n=g+3$ $\to$ $\deg(\mathcal{T}) = 2^g$.

Thm (Cela-Pandharipande-Schmitt '21)

via Hurwitz spaces

When $r=1$, d, n arbitrary:

$$\deg(\tau) = 2^g - \underline{\text{something else}}$$

vanishes when $d \geq g+1$.

Thm (Farkas-L.) $\displaystyle\int_{Gr(2,r+1)} \sigma_1^g \cdot \sum_{i+j=n-3} \sigma_i \sigma_j = 2^g - \text{something}$

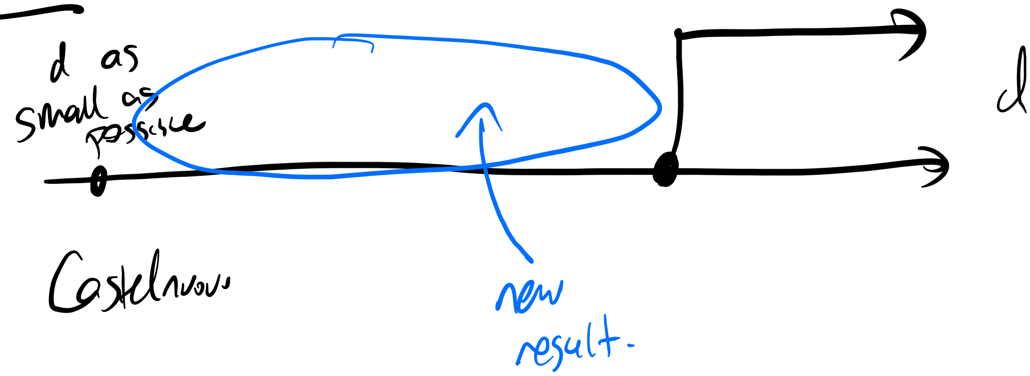
(a) when $r=1$,

$\deg(\tau) =$ formula from Thm A.

(b) when r is arbitrary and $d \geq rg+r$,

$$\deg(\tau) = (r+1)^g.$$

Thm A: interpolates between GW.



Ideas in proof

① Degeneration to genus 0.

$f: C \xrightarrow{\deg d} \mathbb{P}^r$ $\rightsquigarrow$ $f: \mathbb{P}^1 \xrightarrow{\deg d} \mathbb{P}^r$

$f(p_i) = x_i$ $\rightsquigarrow$ $f(p_i) = x_i$.

f ramified at the pts $q_1, \ldots, q_g$.

[$\Leftarrow$) The hyperplane of sections defining f that vanish at q_j actually vanishes to order 2.]

$\Leftrightarrow$) Schubert condition of class $\sigma_{1,r}$ on $Gr(r+1, H^0(\mathbb{P}^1, \mathcal{O}(d)))$.

$$\rightsquigarrow \sigma_{1,r}^{g}$$

Needed Ingredients

① Limit Linear Series (Eisenbud–Harris).

② Space of Complete Collineations avoids

Contributions from "maps with basepoints" by blowing up loci of <u>degenerate maps</u> (maps with image contained in a hyperplane.)

Reduce to the following problem:

$$\begin{tikzcd}
\left\{ \begin{array}{c} f : \mathbb{P}^1 \to \mathbb{P}^r \ \deg d \\ \text{with } f(p_i) = x_i \end{array} \right\} \arrow[d, dashed] & \left\{ \begin{array}{c} \text{Space of} \\ \text{complete} \\ \text{collineations} \end{array} \right\} \arrow[l] \arrow[dl] \\
Gr(r+1, H^0(\mathcal{O}(d))) &
\end{tikzcd}$$

What is the class of the closure of the image,

$$Y_{r,n,d} \subset Gr(r+1, d+1) \ ?$$

Quick cram to finish

(1) reduce to the case $d = n-1$.

(2) $Y_{r,n,n-1} \subset Gr(r+1, d+1)$ is a generic T-orbit closure.

$$Gr(r+1, d+1) = Gr(r+1, \mathbb{C}^n) \circlearrowleft (\mathbb{C}^\times)^n$$

Classes are understood:

80s • Klyachko

'05 • Anderson-Tymoczko $(Fl(n))$

'12 • Berget-Fink

$$\hookrightarrow \sum_{\lambda \subset (n-r-2)} \sigma_\lambda \sigma_{\overline{\lambda}} = [Y_{r,n,n-1}]$$

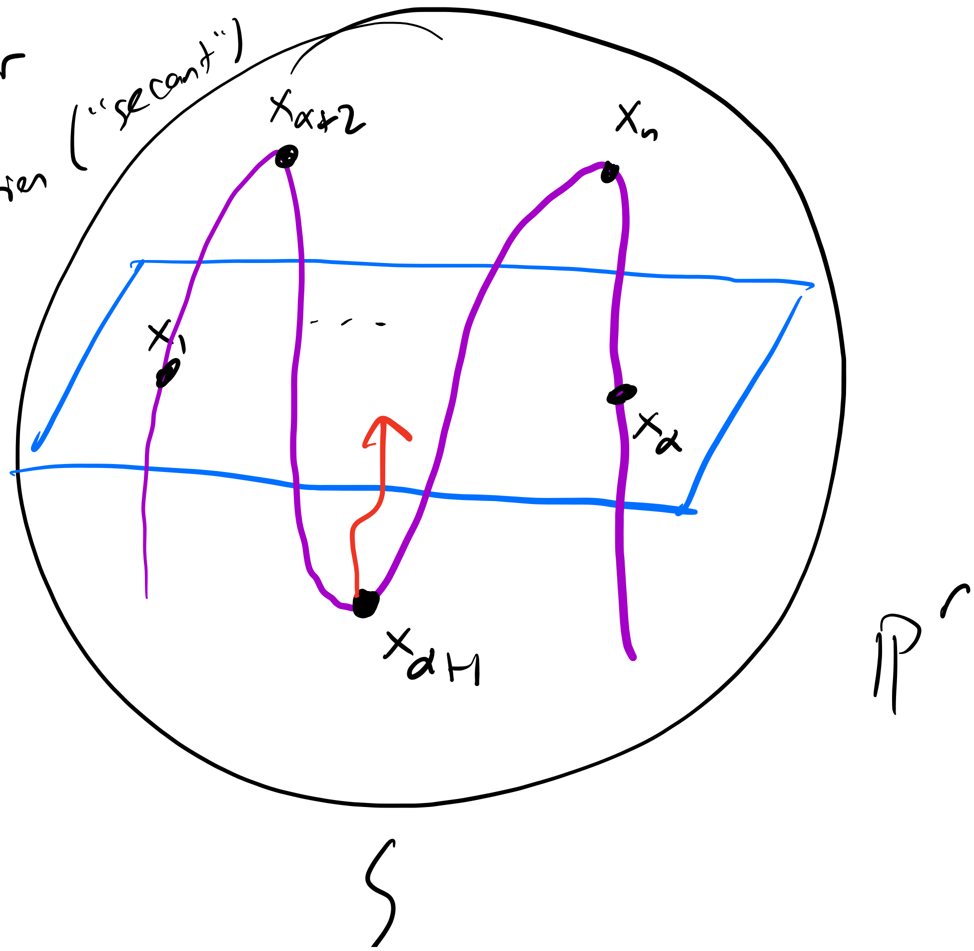
② Class of $Y_{r,n,d}$ by degeneration of the points $x_i \in \mathbb{P}^r$. [recovers orbit closure formulas].

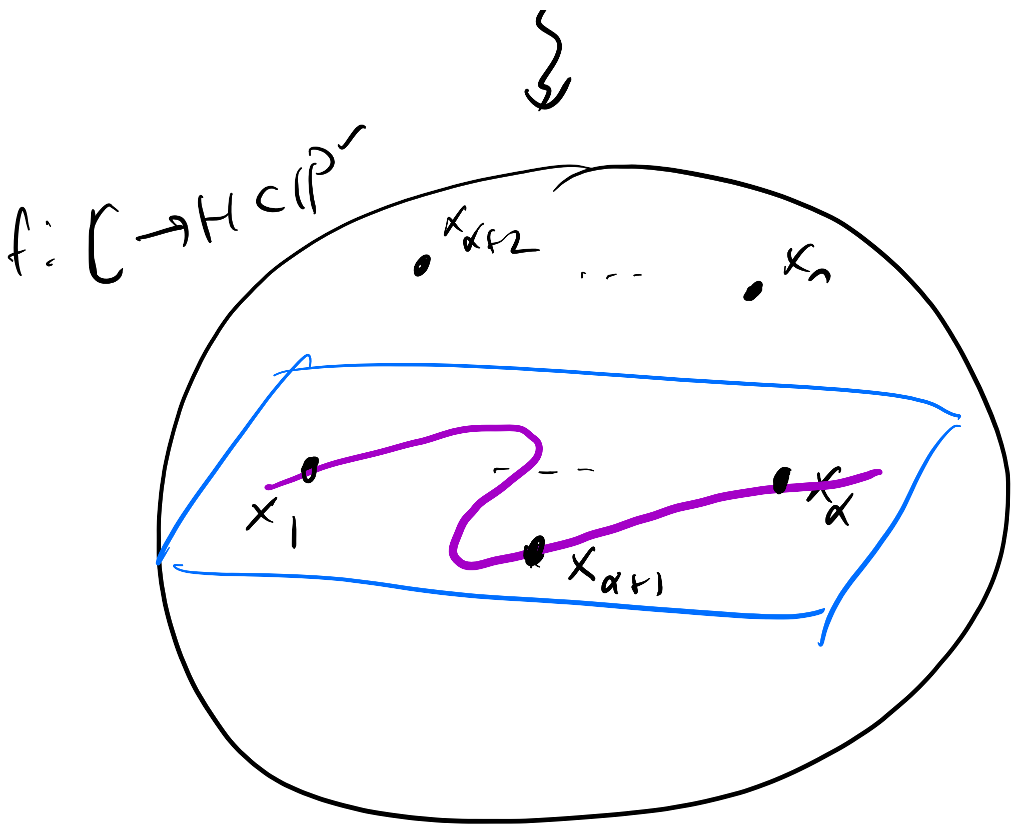
$f: \mathbb{P}^1 \to \mathbb{P}^r \qquad f(p_i) = x_i$.

Idea: Move the points $x_1, x_2, \ldots$ one by one onto a hyperplane in $\mathbb{P}^r$, study degeneration of f.

$f: C \to \mathbb{P}^r$

$\exists$ a section ("secant") that vanishes along the divisor $p_1 + \cdots + p_a$.





"BP condition"

- $p_{\alpha+2}, \dots, p_n$ become basepoints of C.

- the limit of the linear series underlying f still contains a non-zero secant vanishing along $p_1 + \dots + p_\alpha$. "secancy condition"

Repeat: $X_1, \dots, X_{\alpha+1}$ into codim 2 subspace, iterate.

in the end:

(a) nested sequence of B? conditions $\overline{\sigma}_\lambda$

(b) nested sequence of secancy conditions $\overline{\sigma}_\lambda$.