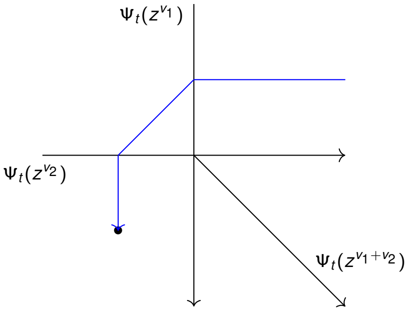
Quantum theta bases

Travis Mandel



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- Also expect canonical bases for *quantum* cluster algebras (Berenstein-Zelevinsky, Fock-Goncharov).
- [Davison-M]: GHKK arguments + DT theory $\rightsquigarrow$ Quantum theta bases.

Cluster algebras

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Applications:

- Representation theory and quantum groups;
- (Higher) Teichmüller theory and Poisson geometry;
- DT–theory and quiver representations;
- Mirror symmetry;
- ...

Examples:

- Semisimple Lie groups;
- Grassmannians, other partial flag varieties, and Schubert varieties;
- Higher Teichmüller spaces;
- All log Calabi-Yau surfaces;
- ...

Cluster varieties

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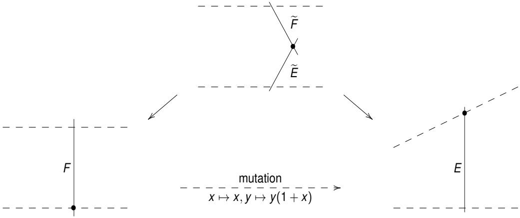

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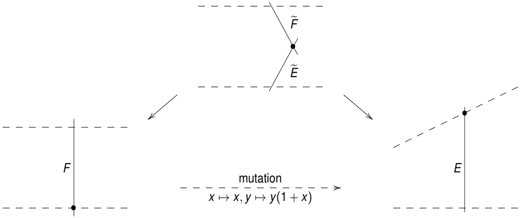

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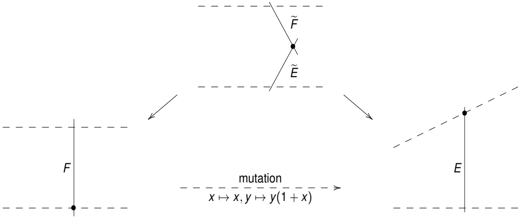

- Gross-Hacking-Keel: Interpret mutations as a blow-up followed by a blow-down.
- Upper cluster algebra — ring of global regular functions on the cluster variety.
- Cluster algebra — subring generated by the "cluster monomials," i.e., elements which are monomials in some cluster.

Seeds

A (skew-symmetric) **seed** is the data $S = (N, I, E, B)$, where

- N is a finite-rank lattice;
- $E = \{e_i | i \in I\}$ is part of a basis for N, indexed by I;
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- Let $v_i := B(e_i, \cdot) \in M$.
- Let $M^{\oplus}$ denote the positive span of the v_i's.
- To quantize, need a $\mathbb{Z}$-valued skew-symmetric form Λ on M such that

$$\Lambda(\cdot, v_i) = e_i \qquad \text{for all } i \in I.$$

Quantum tori

Define the **classical torus algebra** (coordinate ring on $N \otimes \mathbb{C}^*$):

$$\mathcal{A}^S := \mathbb{C}[M] := \mathbb{C}[z^u | u \in M]/\langle z^u z^v = z^{u+v}\rangle.$$

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Use Λ to define the **quantum torus algebra**

$$\mathcal{A}_t^S : \mathbb{C}_t^\Lambda[M] := \mathbb{C}[t^{\pm 1}][z^u | u \in M]/\langle z^u z^v = t^{\Lambda(u,v)} z^{u+v} \rangle.$$

Seed mutation

Given S and $j \in I$, define a new seed $\mu_j(S)$ by replacing each e_i with

$$e_i' := \mu_j(e_i) := \begin{cases} e_i + \max(0, B(e_i, e_j))e_j & \text{if } i \neq j \\ -e_j & \text{if } i = j. \end{cases}$$

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Classical cluster mutation:

$\mu_j^{\mathcal{A}} : \mathcal{A}^S \dashrightarrow \mathcal{A}^{\mu_j(S)}$, $z^m \mapsto z^m(1 + z^{v_j})^{\langle e_j, m \rangle}$.

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The cluster variety $\mathcal{A}$ is constructed by gluing algebraic tori via all possible sequences of mutations as above.

Quantum binomial coefficients

- For $k \in \mathbb{Z}_{\geq 0}$, define

$$[k]_t := \frac{t^k - t^{-k}}{t - t^{-1}} = t^{-k+1} + t^{-k+3} + \ldots + t^{k-3} + t^{k-1} \in \mathbb{C}[t^{\pm 1}].$$

  Note $\lim_{t \to 1} [k]_t = k$.

- Define

$$[k]_t! := [k]_t [k-1]_t \cdots [2]_t [1]_t.$$

- For $r, k \in \mathbb{Z}_{\geq 0}$, $r \geq k$, define

$$\binom{r}{k}_t := \frac{[r]_t!}{[k]_t! [r-k]_t!}$$

Quantum cluster mutation

Recall $\mathcal{A}_t^S := \mathbb{C}_t^\Lambda[M]$, and recall

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For quantum mutation $\mu_j^{\mathcal{A}_t} : \mathcal{A}_t^S \dashrightarrow \mathcal{A}_t^{\mu_j(S)}$, say that for $\langle e_j, m\rangle \geq 0$, we have

$$\mu_j^{\mathcal{A}_t} : z^m \mapsto \sum_{k=0}^{\langle e_j, m\rangle} \binom{\langle e_j, m\rangle}{k}_t z^{m+kv_j}.$$

[Berenstein-Zelevinsky]

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[Berenstein-Zelevinsky]

Typically defined in terms of conjugation by a quantum dilogarithm:
$\mu_j^{\mathcal{A}_t}(z^n) := \Psi_t(z^{v_j}) z^n \Psi_t(z^{v_j})^{-1}$ where

$$-\operatorname{Li}(-x;t) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k[k]_t} x^k,$$

$$\Psi_t(z^{v_j}) := \exp(-\operatorname{Li}(-z^{v_j};t)).$$

Quantum cluster varieties

- Given $\vec{\jmath} = (j_1, \ldots, j_k) \in I^k$, let
$$\mu_{\vec{\jmath}} = \mu_{j_k} \circ \cdots \circ \mu_{j_1}.$$
Denote $S_{\vec{\jmath}} := \mu_{\vec{\jmath}}(S)$.
- Similarly define $\mu_{\vec{\jmath}}^{\mathcal{A}_t}$.

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  Denote $S_{\vec{\jmath}} := \mu_{\vec{\jmath}}(S)$.
- Similarly define $\mu_{\vec{\jmath}}^{\mathcal{A}_t}$.
- Define

$$\mathcal{A}_t^{\mathrm{up}} := \left\{ f \in \mathcal{A}_t^S \middle| \ \mu_{\vec{\jmath}}^{\mathcal{A}_t}(f) \in \mathcal{A}_t^{S_{\vec{\jmath}}} \text{ for all tuples } \vec{\jmath} \text{ of indices in } I \right\}.$$

Positivity

Let $f \in \mathcal{A}_t^{\mathrm{up}} \setminus \{0\}$.

- f is **universally positive** if each $\mu_{\vec{\jmath}}^{\mathcal{A}_t}(f)$ has positive integer coefficients.
- f is **atomic** if it is universally positive, but is not a sum of two other universally positive elements.

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- f is **atomic** if it is universally positive, but is not a sum of two other universally positive elements.
- A basis is **strongly positive** if the structure constants are non-negative.

The Fock-Goncharov conjecture

Conjecture (Fock-Goncharov)

The atomic elements are indexed by M and form a basis for $\mathcal{A}_t^{\mathrm{up}}$ *which includes all the quantum cluster monomials.*

Note: atomic $\Longrightarrow$ strong positivity $\Longrightarrow$ universal positivity.

The Fock-Goncharov conjecture

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Note: atomic $\implies$ strong positivity $\implies$ universal positivity.

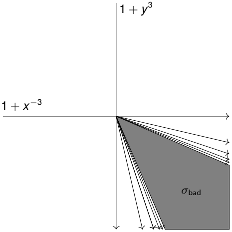
This conjecture is not quite right:

- Lee-Li-Zelevinsky: The atomic elements may be linearly dependent.
- Gross-Hacking-Keel: $\mathcal{X}^{\mathrm{up}}$ is often just $\mathbb{C}$ and so cannot have a basis indexed by N.

Modified Fock-Goncharov conjectures

Corrections to the Fock-Goncharov conjecture:

- Need more charts when defining universal positivity, not just the clusters.



- Sometimes need to work with a formal completion:

$$\widehat{\mathcal{A}}_t := \mathbb{C}_t^{\Lambda}[M] \otimes_{\mathbb{C}_t^{\Lambda}[M^{\oplus}]} \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!].$$

Then the basis should only be a "topological basis."

Theorem: Quantum theta bases

Theorem (Davison-M)

Subject to the above modifications, the quantum Fock-Goncharov conjectures are true.

- In the classical limit, we recover [Gross-Hacking-Keel-Kontsevich].

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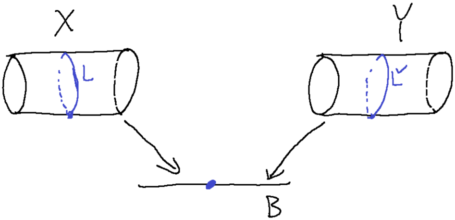
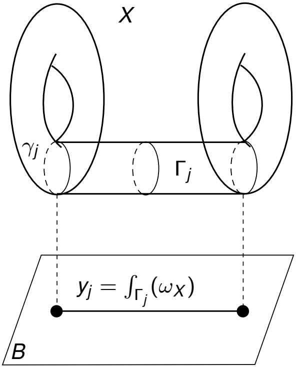
- In the classical limit, we recover [Gross-Hacking-Keel-Kontsevich].
- The "full Fock-Goncharov conjecture" holds in nice situations.

SYZ Mirror Symmetry

The geometric intuition behind [GHKK] comes from SYZ mirror symmetry.

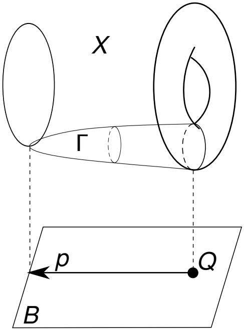


Local coordinates from cylinders



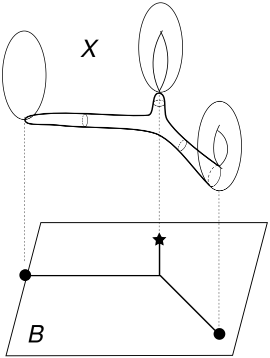
- Mirror Y locally looks like $TB/T_{\mathbb{Z}}B$.
- Local algebraic coordinates on Y: $z_j := \exp[2\pi i(dy_j + iy_j)]$.

Global coordinates from holomorphic disks



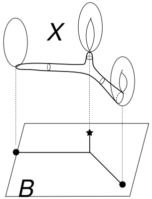
$$\vartheta_{p,Q} := \sum_{\Gamma} \exp(2\pi i(dy_{\Gamma} + iy_{\Gamma}))$$

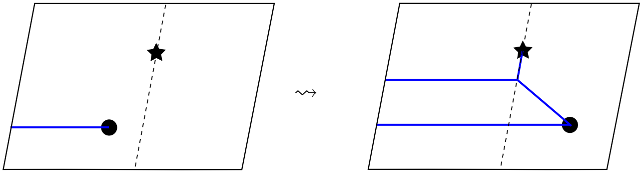
More complicated holomorphic disks



Singular SYZ fibers result in more complicated disks.

Wall-crossing

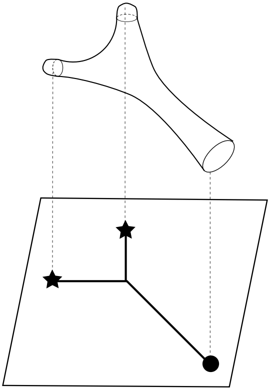




- This leads to “wall-crossing,” or non-trivial transition maps between different local coordinate systems.
- E.g., $(\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^2, \quad x^{-1} \mapsto x^{-1}(1+y).$

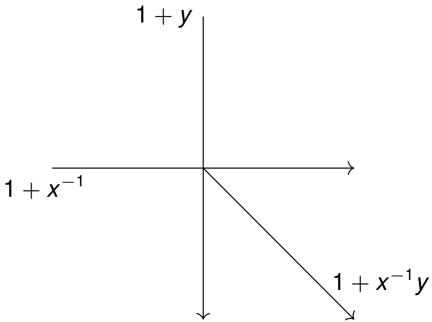
Scattering

Initial walls can interact to form new walls.



Scattering diagrams

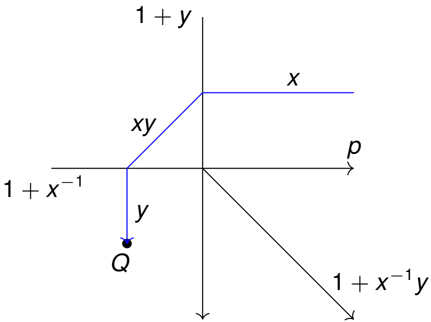
The data of these walls is encoded in a “scattering diagram.”



Walls labelled with functions indicating the corresponding transition functions.

Broken lines

Broken lines are essentially tropical versions of the holomorphic disks used to construct the theta functions.



Scattering diagrams

- A **scattering diagram** $\mathfrak{D}$ is a collection of codimension one walls $\mathfrak{d}$ in $M_{\mathbb{R}}$ with attached elements $f_{\mathfrak{d}} = 1 + \sum_{k=1}^{\infty} c_k z^{kv_{\mathfrak{d}}} \in \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!]$, where $\mathfrak{d} \subset v_{\mathfrak{d}}^{\Lambda\perp}$.

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- Path $\gamma \subset M_{\mathbb{R}} \rightsquigarrow$ **path-ordered product** $\theta_{\gamma,\mathfrak{D}} : \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!] \xrightarrow{\sim} \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!]$:
  - Whenever γ crosses a wall, conjugate by the function $f_{\mathfrak{d}}$ attached to the wall (or its inverse).
  - $\mathfrak{D}$ is called **consistent** if $\theta_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ.
  - $(\mathfrak{d}, f_{\mathfrak{d}})$ called **incoming** if $v_{\mathfrak{d}} \in \mathfrak{d}$.

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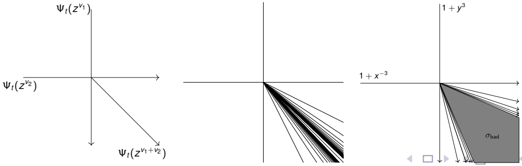
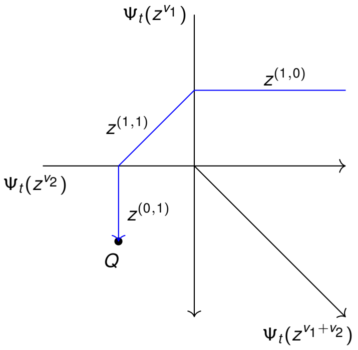
  - $(\mathfrak{d}, f_{\mathfrak{d}})$ called **incoming** if $v_{\mathfrak{d}} \in \mathfrak{d}$.
- Let
$$\mathfrak{D}_{\mathrm{in}}^{\mathcal{A}} := \{(e_i^{\perp}, \Psi_t(z^{v_i})) | i \in I\}.$$
- This uniquely determines a consistent scattering diagram $\mathfrak{D}^{\mathcal{A}} = \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}}^{\mathcal{A}})$ with $\mathfrak{D}_{\mathrm{in}}^{\mathcal{A}}$ as the only incoming walls.



Broken lines

Broken line with ends (p, Q), $p \in M$, Q generic in $M_{\mathbb{R}}$ — a piecewise-straight path $\gamma : (-\infty, 0] \to M_{\mathbb{R}}$, bending only at walls, with a monomial $a_i z^{p_i} \in \mathbb{C}_t[M]$ attached to each straight segment, such that:

- The first attached monomial is z^p,
- $\gamma(0) = Q$,
- $p_i = -\gamma_i'$
- $a_{i+1} z^{p_{i+1}}$ is a term in $\theta_{\mathfrak{d}_i}(a_i z^{p_i})$.



Theta functions

For $p \in M$, $Q \in M_{\mathbb{R}}$, define

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

$a_\gamma z^{m_\gamma}$:= monomial attached to the last straight segment of γ.

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For consistent scattering diagrams, different choices of Q are related by path-ordered product.

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Lemma (Carl-Pumperla-Siebert, M)

For consistent scattering diagrams, different choices of Q are related by path-ordered product. Interpret this as saying that we actually have a single global function ϑ_p for each p, and then the $\vartheta_{p,Q}$'s are expansions in different local cooridnate systems.

Positivity of the scattering diagram

Theorem (Davison-M)

Up to equivalence, every scattering function of $\mathfrak{D}^{\mathcal{A}}$ has the form $\mathbb{E}(-p(t)z^v)$ for some $p(t) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}]$.

- Here, $\mathbb{E}$ is the "plethystic exponential," an algebraization of the graded symmetric product.
- $\mathbb{E}(-z^v) = \Psi_t(z^v)$.
- Generally, $\mathbb{E}(-p(t)z^v)$ factors as a product of $\Psi_{t^k}(t^a z^{kv})$ for $k \in \mathbb{Z}_{\geq 1}$, $a \in \mathbb{Z}$.

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$$\begin{aligned} \text{Theorem} &\implies \text{Positivity of broken lines} \\ &\implies \text{Universal and strong positivity and atomicity.} \end{aligned}$$

Quiver representations and stability conditions

- Let Q be a quiver without oriented 2-cycles.
- $Q \rightsquigarrow$ a seed with $N = \mathbb{Z}^{Q_0}$, $I = Q_0$, $E = \{e_i | i \in I\}$, $B =$ adjacency matrix for Q.

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- $\zeta \in M_{\mathbb{R}} \rightsquigarrow$ stability conditions for $\mathrm{rep}(Q)$: Say V is ζ-semistable if:
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Application in preparation (joint with F. Qin)

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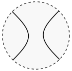
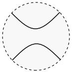
 $= t$  $+t^{-1}$ 

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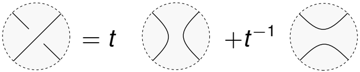

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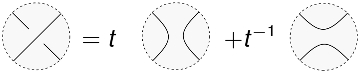

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- Theorem [M-Qin]: Quantum bracelet bases are quantum theta bases.