29/12/24 # The Defect of a cubic 3fold

- joint with Sasha Viktorova

X = cubic 3fold w/ isolated sings $/\mathbb{C}$.
not a cone over a cubic surface.

Defect: $\sigma(X) = \operatorname{rk}\left(\mathrm{Weil}(X)/\mathrm{Cart}(X)\right)$

* measures failure of $\mathbb{Q}$-factoriality

Cheltsov: nodal cubic 3folds are $\mathbb{Q}$-factorial if # nodes < 4.

If # nodes $= 4$ all nodes may lie on a plane $P \subset X$
↖ Weil div, not Cart.

nodes $= 6$ in general position
$\iff$ X being determinental.
Hasset-Tschinkel : contain rational
Dolgachev normal cubic scroll.
↗ Weil div not Cart.

$\Rightarrow \sigma(X) > 0$.

* measures failure of Poincaré duality:
Steenbrink + Namikawa: X normal, proj 3fold, isol rational sings, $H^{2,0} = 0$ then

$$\sigma(X) = b_4(X) - b_2(X)$$

* $\sigma(X)$ connected to existence of reducible fibers of Intermediate Jacobian fibration.

$V \subset \mathbb{P}^5$ smooth cubic 4fold.

Donagi Markman

$$\begin{array}{ccc} \pi_U : J_U & \longrightarrow & U \subset (\mathbb{P}^5)^\vee \\ \cup & & \cup \\ J(X_t) & \longrightarrow & t \end{array}$$

U smooth hyperplane sections

• Laza, Saccà, Voisin: $\exists$ HK compactification $\pi : \mathcal{J}_V \longrightarrow (\mathbb{P}^5)^\vee$

V general cubic 4fold: π has irreducible fibers.

• Saccà: all cubic 4folds - not constructive, and π may have reducible fibers.

Brosnan: If $\exists\ H \cap V = X$ with $\sigma(X) > 0$, then fiber over H is reducible

irr components $\geq \sigma(X) + 1$.

3 Main Goals:

1. Understand which cubic 3folds $\sigma(X) > 0$.

2. Identify generators of $\mathrm{Weil}(X) / \mathrm{Cart}(X)$.
   ↳ in turn: provide geometric criteria for when a cubic 4fold V has a hyperplane sect with positive defect.

3. Compute cohomology - Mixed Hodge Str on $H^3(X, \mathbb{Z})$.

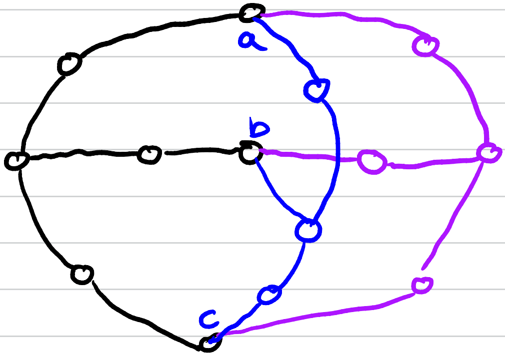
Sings of Cubic 3fold.

* all possible combination isolated sings are classified by Viktorova

Thm (Viktorova): 204 possible combinations.
- ADE sings occurs <=> union of Dynkin diagram is $10A_1$, $5A_2$ or an induced subgraph of:

remove a, b, c.

$3D_4$



3 copies of $\tilde{E}_6$

- worse than ADE do appear.
- maximal ADE : $10A_1$, A_{11},

$D_8 + A_3$, $E_7 + A_2 + A_1$, $E_8 + A_2$, ...

Tools to compute Defect.

① Projection method : $q = [0:0:\dots:1] \in \mathrm{Sing}(X)$

$$X : \quad x_4 f_2(x_0, \dots, x_3) + f_3(x_0, \dots, x_3) = 0$$

$$\begin{tikzcd} \mathrm{Bl}_q X \arrow[d] \arrow[dr, "\phi"] & \\ X \arrow[r, dashed] & \mathbb{P}^3 \end{tikzcd}$$

$Q = \{f_2 = 0\}$
$S = \{f_3 = 0\}$.
$C_q = S \cap Q$

$q = A_1 \Rightarrow Q$ smooth
A_n 
$D, E \Rightarrow Q = P_1 \cup P_2$
worse $\Rightarrow$ double plane.

C_q parametises lines in X that pass through q.

② $\mathrm{Bl}_q X \cong \mathrm{Bl}_{C_q} \mathbb{P}^3$

Slogan : C_q governs the geometry X.

Sings of X are mirrored by C_q.

Wall : • If C_q has sing pt p of type T away $\mathrm{Sing}(Q)$ then X has sing of type T on line $\langle p, q \rangle$
• If C_q has sing of type T at $p \in C_q \cap \mathrm{Sing}(Q)$ then $\mathrm{Bl}_q X$ has sing pt of type T on $\phi_E^{-1}(p)$.

Thm 1 (M-Viktorova): $X \subset \mathbb{P}^4$ cubic isol sings (not cone).
Let $k = \#$ irr components of C_q
Then

$$\sigma(X) = \begin{cases} k-1 & \text{when } Q \neq P_1 \cup P_2 \\ k-2 & \text{when } Q = P_1 \cup P_2 \end{cases}$$

Idea of proof: two discriminant squares.

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[d] & Bl_q X \arrow[d] \\ q \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, hook] \arrow[d] & Bl_q X \cong Bl_{C_q} \mathbb{P}^3 \arrow[d] \\ C_q \arrow[r, hook] & \mathbb{P}^3 \end{tikzcd}$$

$\mathbb{P}^1$ bundle over C_q

Cor (M-Viktorova): $\sigma(X) \leq 6$
- $\sigma(X) = 6 \iff X$ cone over cubic surface.
- $\sigma(X) = 5 \iff X$ is Segre cubic ($10 A_1$).

- obtain classification of which cubic 3folds are $\mathbb{Q}$-factorial.

What are generators of $\mathrm{Weil}(X)/\mathrm{Cart}(X)$?

Thm 2 (M-Viktorova): X cubic isol. sings.
Then $\sigma(X) > 0 \iff X$ contains a plane or rational normal cubic scroll.

Idea: $\sigma(X) > 0 \iff C_q$ reducible.

Possible Components

① line $\Rightarrow$ plane immediately.
② plane conic $\Rightarrow$ plane

③ C_q = union of twisted cubics lie on $Q \cap S$.

$X: \quad \det M = 0$

$\phi : X \dashrightarrow \mathbb{P}^2$
$p \mapsto [a,b,c)$ whe $M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$.

$\phi^{-1}(\ell)$ cubic scroll contained in X.

* $V \subset \mathbb{P}^5$ cubic 4fold, contains a plane or cubicscroll,

$\mathcal{J}_V \to (\mathbb{P}^5)^\vee$ $\exists$ reducible fibers $\leftrightarrow$ cubic 3folds w/ $\sigma(X) > 0$.

<u>Q</u>: $V \subset \mathbb{P}^5$ cubic 4fold assume that $\sigma(X) = 0$ for every $X \cap H$?

<u>Int Jacobians for singular X</u>

Mumfords Prym description: $\ell \subset X$ general. avoid sing pts.

$$\begin{array}{c} Bl_\ell X \\ \downarrow \\ \mathbb{P}^2 \supset \underset{\text{quintic}}{D_\ell} \xleftarrow{2:1} \tilde{D}_\ell \end{array}$$

smooth case: $\tilde{D}_\ell$ irreducible étale double cover

$J(X) \cong \mathrm{Prym}(\tilde{D}_\ell, D_\ell)$.
$\qquad\qquad \shortparallel$
$\tau \circlearrowright J(\tilde{D}_\ell) \qquad \mathrm{Fix}(\tau - \mathrm{id})_0$.

Casalaina-Martin, Laza: defined very good line
X sing, $\ell \subset X$ is very good

$\Rightarrow \tilde{D}_\ell \to D_\ell$ is étale, irreducible curves
sings of $D_\ell \Longleftrightarrow$ Sings of X

Laza-Saccà Voisin: prove X with $\mu(X) \leq 5$,
then a general line is a very good line.
$\Rightarrow \overline{J(X)}$

$F(X)$ irreducible.

Thm 3 (M-Viktorova): X cubic 3fold
then $\exists$ very good line $\Longleftrightarrow \sigma(X) = 0$
$\Longleftrightarrow F(X)$ irreducible

A_{10}, $\sigma(X) = 0$.
(I said A_{11}, but meant A_{10}).

<u>Cohomology of X.</u>

Rough Thm 4 (M-Vik): Hodge numbers of MHS on $H^3(X)$ are determined by
(global) $\sigma(X)$ invariant + two local invariants
sings of X.

- $(Y, E) \to (X, q)$ log resol $Y \setminus E \cong X \setminus q$.
E snc

• $b^{p,q} := \dim H^q(Y, \Omega^p_Y(\log E)(-E))$ $p \geq 0$, $q > 0$.

$\Rightarrow$ for cubic, only non zero $b^{1,1}$.

• Link invariant $\ell^{1,1}$

$$\mu = 2b^{1,1} + \ell^{1,1}.$$

Thm: $\underline{h}^{p,q} = \dim Gr^p_F H^{p+q}$

For cubic 3fold $\underline{h}^{1,2} = 5 - \sum b^{1,1}$

$$\underline{h}^{2,1} = 5 - (L - \sigma) - B.$$

$H^3(X)$ Hodge diamond.

weight 4 $\quad 0$

weight 3 $\quad 5-(L-\sigma)-B \quad 5-(L-\sigma)-B$

$\quad L-\sigma$

X cubic $D_4 + 2A_1$ sings.

$q = D_4$ pt.

$Q = P_1 \cup P_2$.

$C_q = C_1 \cup C_2$ plane cubics.

$2A_1$ sings: • C_i nodal cubics. $\sigma(X) = 0$.

• C_1 smooth cubic, $C_2 =$ line $\cup$ conic

$\sigma(X) = 1$.