Abelianisation of Meromorphic Connections

based on arXiv:1902.03384 and work in progress with Marco Gualtieri

Nikita Nikolaev

The University Of Sheffield.

16 September 2021

Online Nottingham Algebraic Geometry Seminar

- Let $(\mathsf{X}, \mathsf{D}) :=$ compact Riemann surface with divisor

  Require $\mathsf{D} \neq \varnothing$; and $|\mathsf{D}| \geqslant 3$ if $\mathsf{X} \cong \mathbb{P}^1$

- Goal: develop a correspondence between flat vector bundles on X and flat line bundles on an appropriate cover.
- i.e., express the analytic complexity of flat vector bundles in terms of algebraic geometry of complex curves.

- Procedure proposed by [Gaiotto-Moore-Neitzke] and [Neitzke-Hollands]. Intimately related to exact WKB analysis.

- Recall: ***Higgs bundle*** on (X, D) is $(\mathcal{E}, \phi)$ where
  - $\mathcal{E}$ = holomorphic vector bundle on X;
  - $\phi = \mathcal{O}_{\mathsf{X}}$-linear map $\mathcal{E} \to \omega_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E}$ (i.e., twisted endomorphism a.k.a ***Higgs field***).

1. Extract ***spectral data*** := coefficients of characteristic polynomial of ϕ:

$$\mathfrak{s} = (\mathfrak{s}_1, \ldots, \mathfrak{s}_n) \in \bigoplus_{i=1}^{n} \mathsf{H}^0_{\mathsf{X}}\left(\omega_{\mathsf{X}}(\mathsf{D})^{\otimes i}\right)$$

2. Twisted cotangent bundle $\pi : \mathsf{T}^\vee_{\mathsf{X}}(\mathsf{D}) := \mathsf{tot}\left(\omega_{\mathsf{X}}(\mathsf{D})\right) \to \mathsf{X}$ has tautological one-form η. The characteristic polynomial is $\chi := \eta^n + \mathfrak{s}_1 \eta^{n-1} + \cdots + \mathfrak{s}_n \in \mathsf{H}^0_{\mathsf{T}^\vee_{\mathsf{X}}(\mathsf{D})}\left(\pi^* \omega_{\mathsf{X}}(\mathsf{D})^{\otimes n}\right)$
3. Get ***spectral curve*** $\Sigma := \mathrm{Zero}(\chi) \hookrightarrow \mathsf{T}^\vee_{\mathsf{X}}(\mathsf{D})$.

- For generic $\mathfrak{s}$, Σ is smooth and $\pi : \Sigma \to \mathsf{X}$ is $n : 1$ cover with simple ramification.
- $g_\Sigma = n^2(g_{\mathsf{X}} - 1) + \frac{1}{2} n(n-1)|\mathsf{D}| + 1$.
- η defines meromorphic one-form on Σ with poles along $\Delta := \pi^* \mathsf{D}$.

Theorem (Hitchin, Beauville-Narasimhan-Ramanan)

$$\left\{ \begin{matrix} \textit{Higgs bundles } (\mathcal{E}, \phi) \textit{ on } (\mathsf{X}, \mathsf{D}) \\ \textit{with spectral data } \mathfrak{s} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \textit{line bundles } \mathcal{L} \\ \textit{on } \Sigma \end{matrix} \right\} = \left\{ \begin{matrix} \textit{Higgs line bundles} \\ (\mathcal{L}, \eta) \textit{ on } (\Sigma, \Delta) \\ \eta : \mathcal{L} \to \omega_\Sigma(\Delta) \otimes \mathcal{L} \end{matrix} \right\}$$

Construction: $\boxed{\rightarrow}$: $\mathcal{L}$ ":=" $\mathit{ker}(\pi^* \phi - \eta)$; $\boxed{\leftarrow}$: $\mathcal{E} := \pi_* \mathcal{L}$ and $\phi := \pi_* \eta$.

- Goal: generalise this procedure from Higgs bundles to **flat bundles**.
- ***Meromorphic connection*** on (X, D) is $(\mathcal{E}, \nabla)$ where
  - $\mathcal{E} =$ holomorphic vector bundle on X;
  - $\nabla =$ a $\underline{\mathbb{C}}$-linear map $\mathcal{E} \to \omega_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E}$ satisfying *Leibniz rule*:

$$\nabla(fe) = f\nabla e + \mathrm{d}f \otimes e \qquad \forall f \in \mathcal{O}_{\mathsf{X}}, e \in \mathcal{E}$$

- i.e.: $\nabla = 1^{\text{st}}$-order meromorphic differential operator on sections of $\mathcal{E}$
- Locally, $\nabla \underset{\text{loc}}{=} \mathrm{d} + \phi$ where $\phi =$ Higgs field on $\mathcal{E}$
- If $\mathsf{p} \in \mathsf{D}$ has multiplicity $m \geqslant 1$ and $z(\mathsf{p}) = 0$, then $\nabla \underset{\text{loc}}{=} \mathrm{d} + A(z) z^{-m}\, \mathrm{d}z$ where $A(z) =$ holomorphic matrix
- Locally, the same as a singular ODE $\nabla_{\partial_z} e(z) = \partial_z e(z) + A(z) z^{-m} e(z) = 0$.

- Want some statement of the form

$$\text{“}\left\{\begin{matrix}\text{meromorphic connections}\\ (\mathcal{E}, \nabla) \text{ on } (\mathsf{X}, \mathsf{D})\\ \text{of rank } n\end{matrix}\right\}\text{”} \quad \xleftrightarrow{1:1} \quad \text{“}\left\{\begin{matrix}\text{meromorphic connections}\\ (\mathcal{L}, \partial) \text{ on } (\Sigma, \Delta)\\ \text{of rank } 1\end{matrix}\right\}\text{”}$$

Problems:

1. connections don't have invariant notion of eigenvalues, so what is Σ?
2. connections don't have invariant notion of eigenvectors, so what replaces $\mathcal{L}$?
3. (most serious!) direct image π_* cannot be the right thing because any $\pi_*\partial$ necessarily has non-trivial singularities along branch locus of special type given by the permutation representation of π.

Polar Spectral Data and Levelt Filtrations

- We fix generic **polar spectral data** for connections along D:
  - Restricting $(\mathcal{E}, \nabla)$ to the subscheme $\mathsf{D} \subset \mathsf{X}$ gives $\mathcal{O}_{\mathsf{D}}$-linear map

$$\nabla_{\mathsf{D}} : \mathcal{E}_{\mathsf{D}} \to \omega_{\mathsf{X}}(\mathsf{D})|_{\mathsf{D}} \otimes \mathcal{E}_{\mathsf{D}}$$

  - i.e., $(\mathcal{E}_{\mathsf{D}}, \nabla_{\mathsf{D}})$ = holomorphic Higgs bundle on D (the ***polar Higgs bundle***)
  - Get well-defined ***polar spectral data*** $\mathfrak{s}_{\mathsf{D}} \in \bigoplus_{i=1}^{n} \mathsf{H}^0_{\mathsf{D}}\Big(\omega_{\mathsf{X}}(\mathsf{D})|_{\mathsf{D}}^{\otimes i} \Big)$.
  - For example, ∇_{D} = residue if D is reduced.

- **Main Analytic Fact (existence of *Levelt filtrations*):**
  if $\mathfrak{s}_{\mathsf{D}}$ is generic, $(\mathcal{E}, \nabla)$ is locally filtered near D by growth rates of flat sections:

$$\mathcal{E}_{\mathsf{p}}^{\bullet} = \big(\mathcal{E}_{\mathsf{p}}^{1} \subset \cdots \subset \mathcal{E}_{\mathsf{p}}^{n} = \mathcal{E}_{\mathsf{p}}\big) \qquad \text{with} \qquad \nabla_{\mathsf{p}}(\mathcal{E}_{\mathsf{p}}^{k}) \subset \omega_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E}_{\mathsf{p}}^{k}$$

- Locally near D, we obtain a natural *diagonal* connection

$$\mathrm{gr}\, \mathcal{E}_{\mathsf{p}}^{\bullet} = \mathcal{L}_{\mathsf{p}}^{1} \oplus \cdots \oplus \mathcal{L}_{\mathsf{p}}^{n} \qquad \text{with} \qquad \mathrm{gr}\, \nabla_{\mathsf{p}} = \partial_{\mathsf{p}}^{1} \oplus \cdots \oplus \partial_{\mathsf{p}}^{n}$$

- **Observation**: flat line bundles $(\mathcal{L}_{\mathsf{p}}^{k}, \partial_{\mathsf{p}}^{k})$ are what replaces the spectral line bundle $\mathcal{L}$.

1. Choose any smooth simply ramified cover $\pi : \Sigma \to \mathsf{X}$ of degree n with

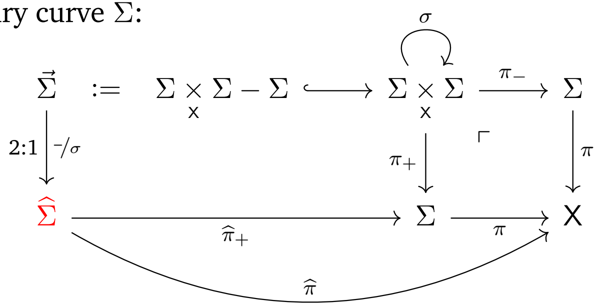
$$\mathsf{Branch}(\pi) \cap \mathsf{D} = \varnothing \qquad \text{and} \qquad g_\Sigma = n^2(g_\mathsf{X} - 1) + \tfrac{1}{2}n(n-1)|\mathsf{D}| + 1.$$

- For example, choose generic spectral data $\mathfrak{s}$ which restricts to $\mathfrak{s}_\mathsf{D}$ along D.
- **Aside**: effectively, we are enriching our setting to the cartoonish diagram

$$\begin{array}{ccc} \mathcal{M}(\mathsf{X}, \mathcal{E}, \nabla, \mathfrak{s}) & \longrightarrow & \mathcal{M}(\mathsf{X}, \mathcal{E}, \nabla) \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathsf{X}, \mathfrak{s}) & \longrightarrow & \mathcal{M}(\mathsf{X}) \end{array}$$

2. Idea: near each $\mathsf{p} \in \mathsf{D}$, lift piece $(\mathcal{L}^k_\mathsf{p}, \partial^k_\mathsf{p})$ of $\mathsf{gr}(\mathcal{E}^\bullet_\mathsf{p}, \mathsf{gr}\,\nabla_\mathsf{p})$ to k-th sheet of Σ above p. Need to choose combinatorial data.

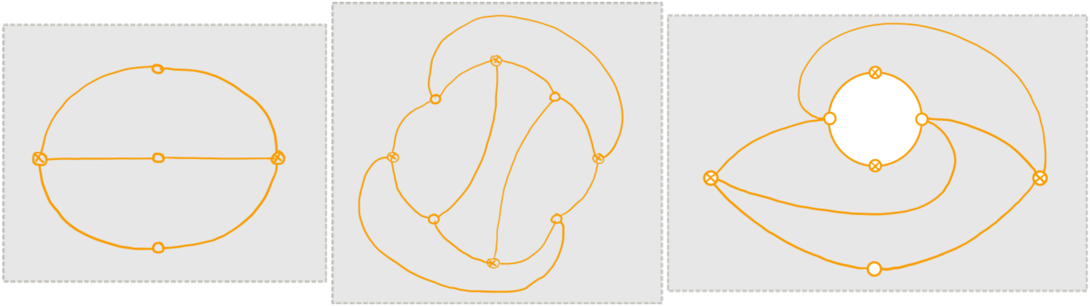
3. Consider an auxiliary curve $\widehat{\Sigma}$:



- Fact: $\widehat{\Sigma}$ is smooth and $\widehat{\pi} : \widehat{\Sigma} \to \mathsf{X}$ is simply ramified with degree $\frac{1}{2}n(n-1)$.
- Also: $\mathsf{Branch}(\widehat{\pi}) = \mathsf{Branch}(\pi)$.
- If $n = 2$, then $\vec{\Sigma} \cong \Sigma$ and $\widehat{\Sigma} \cong \mathsf{X}$.
- If $\Sigma =$ spectral curve with canonical one-form η,
  then $\vec{\Sigma}$ gets a one-form $\vec{\eta} := \pi_+^* \eta - \pi_-^* \eta$,
  and $\widehat{\Sigma}$ gets a quadratic differential $\widehat{q}$ such that $\vec{\eta}^2 = \widehat{q}$.

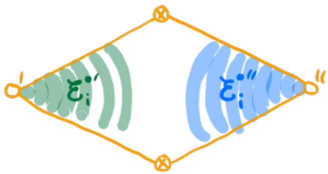
- Let $\widehat{\mathsf{R}} := \mathsf{Ram}(\widehat{\pi})$ and $\widehat{\mathsf{D}} := \widehat{\pi}^* \mathsf{D}$.

4. Choose a ***Stokes graph*** $\widehat{\Gamma}$ on $\widehat{\Sigma}$:= bipartite squaregraph on $\widehat{\Sigma}$ with vertices $(\widehat{\mathsf{R}}, \widehat{\mathsf{D}})$ which is trivalent at each ramification point in $\widehat{\mathsf{R}}$ (plus assumption along $\widehat{\mathsf{D}}$ if D has higher multiplicity).



- For example, if $\widehat{q}$ = quadratic differential on $\widehat{\Sigma}$, then generically $\widehat{\Gamma}$ = locus of critical leaves of the horizontal foliation determined by $\widehat{q}$.
- Amounts to choosing an ideal triangulation of $\widehat{\Sigma}$.

❺ Over each face of $\widehat{\Gamma}$, $(\mathcal{E}, \nabla)$ is filtered in two ways: $\mathcal{E}_i^{\bullet\prime}, \mathcal{E}_i^{\bullet\prime\prime}$ coming from poles $\bigcirc', \bigcirc''$.



We say the Levelt filtrations of $(\mathcal{E}, \nabla)$ are ***generic wrt*** $\widehat{\boldsymbol{\Gamma}}$ if for each face, $\mathcal{E}_i^{\bullet\prime} \pitchfork \mathcal{E}_i^{\bullet\prime\prime}$.

- **Key property**: if the Levelt filtrations of $(\mathcal{E}, \nabla)$ are generic wrt $\widehat{\Gamma}$, then we get canonical isomorphisms over each face:

$$\mathcal{E}_i \cong \operatorname{gr} \mathcal{E}_i^{\bullet\prime} \cong \operatorname{gr} \mathcal{E}_i^{\bullet\prime\prime} \quad \text{and} \quad \nabla_i \simeq \operatorname{gr} \nabla_i^{\bullet\prime} \simeq \operatorname{gr} \nabla_i^{\bullet\prime\prime}$$

❻ Use these identifications as gluing data for a rank-one meromorphic connection $(\mathcal{L}, \partial)$ over Σ with poles along $\Delta := \pi^* \mathsf{D}$ as well as simple poles along $\mathsf{R} := \mathsf{Ram}(\pi)$ with residues $-1/2$. Call these ***branched connections*** on (Σ, Δ).

- Polar spectral data $\mathfrak{p}_\Delta$ of ∂ along Δ is the spectrum of the polar spectral data $\mathfrak{s}_\mathsf{D}$.

Theorem ([N] for $n = 2$; [Gualtieri-N] for $n \geqslant 2$ (to be confirmed))

*There is an equivalence of categories (**abelianisation**):*

$$\left\{ \begin{array}{c} \textit{rank-}n \\ \textit{meromorphic connections} \\ (\mathcal{E}, \nabla) \textit{ on } (\mathsf{X}, \mathsf{D}) \\ \textit{with generic } \mathfrak{s}_{\mathsf{D}} \\ \textit{and generic wrt } \widehat{\Gamma} \end{array} \right\} \xrightarrow[\pi^{\mathrm{ab}}_{\widehat{\Gamma}}]{\sim} \left\{ \begin{array}{c} \textit{rank-one branched} \\ \textit{meromorphic connections} \\ (\mathcal{L}, \partial) \textit{ on } (\Sigma, \Delta) \\ \textit{with generic } \mathfrak{p}_{\Delta} \end{array} \right\}$$

- **Key point:** the inverse equivalence $\pi^{\widehat{\Gamma}}_{\mathrm{ab}}$ (***nonabelianisation***) is a deformation of the direct image functor π_*.
  - An oriented double cover graph $\vec{\Gamma}$ of $\widehat{\Gamma}$ on $\vec{\Sigma}$ determines a canonical cocycle

    $$\alpha \in \check{\mathsf{Z}}^1_{\vec{\Gamma}}\Big(\mathcal{H}om(\pi^*_-, \pi^*_+)\Big)$$

    which induces a unique cocycle

    $$A \text{ ``:='' } \mathbb{1} + \vec{\pi}_* \alpha \in \check{\mathsf{Z}}^1_{\vec{\pi}_* \vec{\Gamma}}\Big(\mathcal{A}ut(\pi_*)\Big)$$

    by completing scattering diagrams.

Abelianisation of $\hbar$-Connections

- This construction extends to ***$\hbar$-connections***: i.e., $(\mathcal{E}, \nabla)$ where
  - $\mathcal{E}$ = holomorphic vector bundle on $\mathsf{X} \times \mathsf{S}$ where $\mathsf{S} \subset \mathbb{C}_\hbar$;
  - ∇ = a $\mathcal{O}_\mathsf{S}$-linear map $\mathcal{E} \to \omega_\mathsf{X}(\mathsf{D}) \otimes \mathcal{E}$ satisfying *$\hbar$-twisted Leibniz rule*:

$$\nabla(fe) = f\nabla e + \hbar \, \mathrm{d}f \otimes e \qquad \forall f \in \mathcal{O}_{\mathsf{X}\times\mathsf{S}}, e \in \mathcal{E}$$

- For fixed $\hbar \neq 0$, these are meromorphic connections in usual sense;
  For $\hbar = 0$, these are Higgs bundles.

Theorem ([N] for $n = 2$; [Gualtieri-N] for $n \geqslant 2$ (to be confirmed))

$$\lim_{\hbar \to 0} \pi^{\mathrm{ab}}_{\widehat{\Gamma}}(\hbar) = \begin{pmatrix} \textit{abelianisation} \\ \textit{of Higgs bundles} \end{pmatrix} \qquad \textit{and} \qquad \lim_{\hbar \to 0} \pi^{\widehat{\Gamma}}_{\mathrm{ab}}(\hbar) = \pi_*$$

- In other words, given $(\mathcal{E}, \nabla)$, let (E, ϕ) be the corresponding Higgs bundle, and (L, η) the corresponding Higgs line bundle. Then have a commutative diagram:

$$\begin{array}{ccc} (\mathcal{E}, \nabla) & \xrightarrow{\hbar \to 0} & (E, \phi) \\ {\scriptstyle \pi^{\mathrm{ab}}_{\widehat{\Gamma}}} \big\downarrow & & \big\downarrow {\scriptstyle \text{abelianisation of Higgs bundles}} \\ (\mathcal{L}, \partial) & \xrightarrow[\hbar \to 0]{} & (L, \eta) \end{array}$$

Thank you for your attention!