A MODULI SPACE OF HOLOMORPHIC SUBMERSIONS

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§ SETTING AND MOTIVATION

$$\begin{array}{c}(Y, H_Y)\\ \downarrow\\ (B, L)\end{array}$$

proper holomorphic submersion of smooth projective varieties

- all fibres are smooth
- $L \to B$ ample line bundle
- $H_Y \to Y$ relatively ample line bundle :
  $H_Y|_{Y_b} \to Y_b$ ample

These fibrations

-> generalise holomorphic vector bundles

-> constitute a way of studying families of projective manifolds

§ SETTING AND MOTIVATION

On vector bundles

$E \longrightarrow B$ holomorphic vector bundle. Assume E is simple: $\Gamma(E, \mathcal{E}nd E) \simeq \mathbb{C}$.

Taking the projectivisation

$X = \mathbb{P}(E) \longrightarrow B$ holomorphic submersion, $\mathcal{O}_{\mathbb{P}(E)}(-1)^\vee \longrightarrow \mathbb{P}(E)$ relatively ample

$\mathcal{O}_{\mathbb{P}(E)}(-1)^\vee \big|_{X_b} = \mathcal{O}_{\mathbb{P}(E)_b}(1)$

Hitchin-Kobayashi correspondence (Narashiman-Seshadri, Donaldson, Uhlenbeck, Yau)

Slope stability $\longleftrightarrow$ $\exists$ Hermite-Einstein connections

(algebro-geometric) (geometric PDE)

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moduli space of holom. vector bundles

GOAL:
- construct a moduli space of smooth fibrations
- generalise the Hermite-Einstein connections to optimal symplectic connections on fibrations with K-semistable fibres

§ SETTING AND MOTIVATION

Example: $E \to C$ holomorphic vector bundle over a curve, ω_C Kähler on C.

<u>Def</u> • slope of E: $\mu(E) = \dfrac{\deg E}{\operatorname{rk} E} =: \dfrac{d}{r}$

where $\deg E = \deg(\Lambda^{\operatorname{rk} E} E) = c_1(E) \cdot [\omega_C]$

• E is stable if $\mu(E) > \mu(F)$ $\forall F \subset E$ subbundle

• E is semistable if $\mu(E) \geq \mu(F)$ $\forall F \subset E$

=> (Mumford) there exists a moduli space of semistable vector bundles with fixed rank and degree, $\mathcal{M}^{ss}(r, d)$. <u>and</u> the moduli space is constructed as a GIT quotient (locally and globally)

§ SETTING AND MOTIVATION

From the "geometric PDE" side: $E \to B$ h on fibres of E

h Hermitian structure induces A_h Chern connection.

<u>Def</u> A_h is Hermite-Einstein if

$$\Lambda_{\omega_B} F_{A_h} = \lambda \mathbb{1}_E \qquad \lambda = \frac{\mu(E)}{\int \omega_c}$$

[Fujiki-Schumacher] there exists a moduli space of vector bundles that admit a Hermite-Einstein connection.

§ SETTING AND MOTIVATION

Back to fibrations $Y \longrightarrow B$:

We need: → a generalisation of Hermite-Einstein connections: optimal symplectic connections. They are solutions to a PDE that are related to the stability of the fibration.

→ stability condition for the fibres: in terms of K-stability

§ SETTING AND MOTIVATION

Back to fibrations $Y \longrightarrow B$:

We need: → a generalisation of Hermite-Einstein connections: optimal symplectic connections. They are solutions to a PDE that are related to the stability of the fibration.

→ Stability condition for the fibres: in terms of K-stability

Another bit of motivation: merge these two pictures

$$\text{Slope stability} \xleftrightarrow{\text{H-K}} \exists\ \text{Hermite-Einstein}$$

$$\text{K-stability} \underset{\text{conj}}{\overset{\text{YTD}}{\longleftrightarrow}} \exists\ \text{Kähler metrics with constant scalar curvature}$$

cscK

$$\mathrm{Scal}(\omega) = \hat{S}$$

§ MAIN RESULT

THEOREM (-) There exists a moduli space that parametrises holomorphic submersions $\pi_Y : (Y, H_Y) \longrightarrow (B, L)$ that

- have discrete relative automorphism group:

$$\mathrm{Aut}(\pi_Y) = \{ g \in \mathrm{Aut}(Y, H_Y) \mid \pi_Y \circ g = \pi_Y \}$$

- admit an optimal symplectic connection.

Such a moduli space is a Hausdorff complex analytic space and it admits a Weil-Petersson type Kähler metric

§ STABILITY OF THE FIBRES

In terms of analytic K-semistability $\longleftrightarrow$ $\exists$ cscK metrics

More precisely:

Assume that $(Y, H_Y) \longrightarrow (B, L)$ degenerates to $(X, H_X) \longrightarrow (B, L)$ such that $\forall b \in B$ (X_b, H_b) has a Kähler metric with constant scalar curvature:

$$\omega_b \in c_1(H_X|_{X_b}) \quad \text{such that} \quad \mathrm{Scal}(\omega_b) = \hat{S}_b$$

1. $\hat{S}_b$ is a topological constant that does not depend on b, because $c_1(H_X|_{X_b})$ is an integer class as cohomology class

2. [Dervan·Sektnan] There exists $\omega \in c_1(H_X)$ s.t. $\omega|_{X_b}$ has constant scalar curvature. ω is RELATIVELY KÄHLER METRIC

§ STABILITY OF THE FIBRES

Degeneration means: $\qquad\qquad S$ = parameter space (disk Δ or $\mathbb{C}$)

$$\boxed{\begin{array}{ccccccccc}
(\mathcal{X},\mathcal{H}) & & (\mathcal{X}_0,\mathcal{H}_0) & \simeq & (X,H_X) & & (\mathcal{X}_s,\mathcal{H}_s) & \simeq & (Y,H_Y) \\
\downarrow & \text{s.t.} & \downarrow & & \downarrow \ \text{rel. cscK} & \text{and} & \downarrow & \forall s\neq 0 & \downarrow \\
B\times S & & B & & B & & B & & B
\end{array}}$$

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How to think of these degenerations:

Let $\mathbb{C}^* \curvearrowright B\times\mathbb{C}$ trivially on B. Then we can realise the degeneration $(\mathcal{X},\mathcal{H}) \to B\times\mathbb{C}$ using a lift of $\mathbb{C}^*$ to $(\mathcal{X},\mathcal{H})$

Philosophically: $(\mathcal{X},\mathcal{H}) \to B\times\mathbb{C}$ is a family of test configurations for the fibres X_b compatible with the fibration structure

§ STABILITY OF THE FIBRES

$$
\begin{tikzcd} (\mathcal{X},\mathcal{H}) \arrow[d] \\ B\times S \end{tikzcd}
\quad \text{s.t.} \quad
\begin{tikzcd} (\mathcal{X}_0,\mathcal{H}_0) \arrow[d] \arrow[r, phantom, "\simeq"] & (X,H_X) \arrow[d] \\ B & B \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} (\mathcal{X}_s,\mathcal{H}_s) \arrow[d] \arrow[r, phantom, "\simeq"] \arrow[dr, phantom, "\forall s\neq 0"] & (Y,H_Y) \arrow[d] \\ B & B \end{tikzcd}
$$

Remark:

1. the fibres of $Y \to B$ are analytically K-semistable

§ STABILITY OF THE FIBRES

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Remark:

1. the fibres of $Y \to B$ are analytically K-semistable
2. A relative version of Ehresmann theorem implies that X and Y are diffeomorphic. Let M = underlying smooth manifold.

   $\Rightarrow$ we can view Y as a deformation of the complex structure of X.

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rel. cscK $\Downarrow$ fibres are K-polystable

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3. $c_1(H_X) = c_1(H_Y) \in H^2(M,\mathbb{Z})$ and $c_1(H_X)$ is of type $(1,1)$ also on Y

$\Rightarrow$ we have $\omega \in c_1(H_X)$ relatively cscK AND we can assume that $\omega \in c_1(H_Y)$ is also relatively Kähler (but no cscK on the fibres)

§ STABILITY OF THE FIBRES

⇒ we fix the smooth structure M and the relatively symplectic form ω

we vary the holomorphic structure:

$X = (M, \omega, J_0) \to B$ $\qquad\qquad$ $Y \simeq \mathfrak{X}_s = (M, \omega, J_s) \longrightarrow B$

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This can be made precise by

THEOREM (-) The deformations of the holomorphic submersion $X \longrightarrow B$ that preserve the projection onto B can be parametrised by an open subset V_π of a finite-dimensional vector space in $\Omega^{0,1}(T^{1,0}_{vert}X)$:

$H^1(T_X)$

$$\Phi : V_\pi \hookrightarrow \mathcal{J}_\pi = \left\{ \begin{array}{l} \text{almost complex structures on } M \text{ compatible} \\ \text{with } \omega \text{ and with the projection onto } B \end{array} \right\}$$

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equivariant w.r.t.

$$K_\pi = \begin{array}{l} \text{biholomorphisms of } X \text{ that commute with } \pi \\ \text{and are fibrewise isometries of the relatively} \\ \text{Kähler form } \omega \end{array}$$

Relative Kuranishi theorem, or Luna slice theorem or Hilbert scheme

§ STABILITY OF THE FIBRES

So we can identify

$$V_\pi \ni 0 \longleftrightarrow \begin{array}{c} X \\ \downarrow \\ B \end{array}$$ rel. cscK K-ps

$$y_s \longleftrightarrow \begin{array}{c} \mathcal{X}_s \\ \downarrow \\ B \end{array} \simeq \begin{array}{c} Y \\ \downarrow \\ B \end{array}$$

and the degeneration $(\mathcal{X}, \mathcal{H}) \to B \times S$ can be realised as an orbit in V_π

$\mathbb{C}^* \cdot y_s$ s.t. $0 \in \overline{\mathbb{C}^* \cdot y_s}$

$\mathbb{C}^* < K_\pi$

Key to construct moduli space: having a degeneration to a relatively cscK fibration is a locally closed property.

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Key to construct moduli space: having a degeneration to a relatively cscK fibration is a locally closed property.

Assume that: $\mathrm{Aut}_0(X_b, H_X|_b)$ are all isomorphic.

Lemma (–) There exists $W \subset V_\pi$ locally closed subvariety such that $\forall w \in W$ the corresponding fibration $Y_w \to B$ admits a degeneration to some $X' \to B$ with cscK fibres.

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Proof : $U \subset B$ open chart. Consider the Kuranishi space of the fibres of $X \to U$ to construct $X' \to U$ locally

$$V_{b_0} \curvearrowleft K_{b_0} = \text{biholomorphic isometries of } (\omega|_{X_{b_0}}, J_b|_{X_{b_0}})$$

$$K_{b_0}^{\mathbb{C}} = \mathrm{Aut}_0(X_{b_0}, H_X|_{b_0})$$

$\Rightarrow$ cscK fibres near X_{b_0} are

- (Szekelyhidi) GIT-polystable points in V_{b_0}
- fixed points by the assumption

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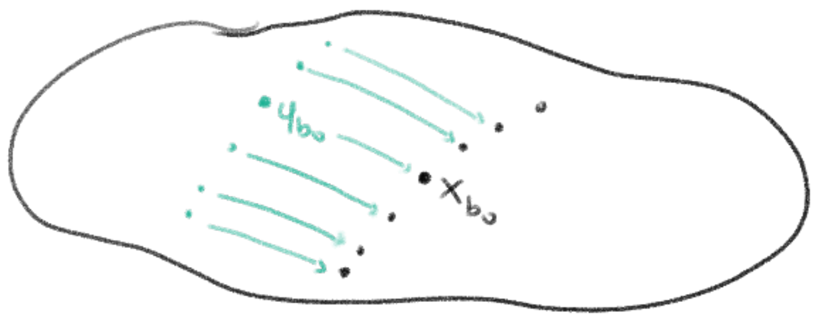
$\Rightarrow \quad \begin{array}{c} X \\ \downarrow \\ U \end{array} \hookrightarrow \{x_b\} \subset V_{b_0}$ fixed $\quad$ and $\quad \begin{array}{c} Y \\ \downarrow \\ U \end{array} \hookrightarrow \{y_b\} \subset V_{b_0}$

s.t. $\quad \overline{K^{\mathbb{C}}_{b_0} \cdot y_b} \ni x_b$



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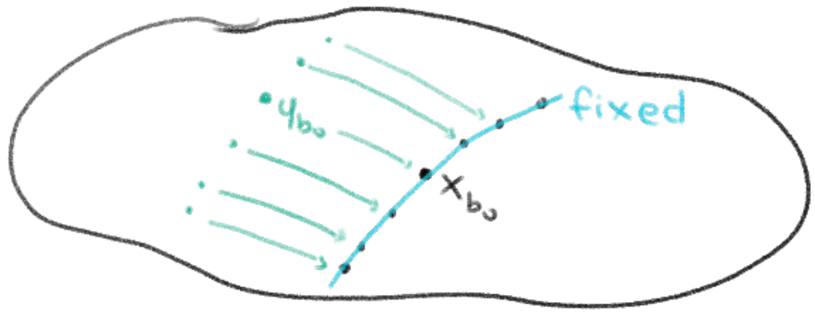

Proof follows from:

Prop (–): There exists an analytic subvariety $V_{b_0}^+$ of V_{b_0} s.t.

$$F : V_{b_0}^+ \longrightarrow V_{b_0}$$
$$y \longmapsto x = \left\{ \begin{array}{l} \text{fixed polystable point in the} \\ \text{closure of the orbit} \end{array} \right\}$$

is holomorphic.

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Lemma (-) There exists $W \subset V_\pi$ locally closed subvariety such that $\forall w \in W$ the corresponding fibration $Y_w \to B$ admits a degeneration to some $X' \to B$ with cscK fibres.

Remark

The proof relies on deep results:

- [Chen-Sun] uniqueness of K-polystable degeneration
- [Székelyhidi], [Brönnle] deformation theory of cscK manifolds
- [Székelyhidi] a K-polystable deformation of a cscK manifold is cscK
- analogy with Białynicki-Birula stratification

§ OPTIMAL SYMPLECTIC CONNECTIONS

DEF Let $Y \to B$ be a holomorphic submersion with K-semistable fibres and let $X \to B$ be a relatively cscK degeneration. A relatively Kähler metric ω is optimal symplectic connection if

$$\boxed{p_E\left(\Delta_{\mathrm{vert}} \Lambda_{\omega_B} F_{\mathcal{H}} + \Lambda_{\omega_B} \rho_{\mathcal{H}} + \lambda \nu\right) = 0}$$

- $\lambda > 0$
- $F_{\mathcal{H}}$ = symplectic curvature of ω
- $\rho = i\partial\bar\partial \log \omega^m$ $\quad m = \mathrm{rel}\dim X \to B$ i.e. ρ = curvature of Hermitian metric induced by ω on $\Lambda^m T^{1,0}_{\mathrm{vert}} X = -K_{X/B}$
- Curvature quantity of deformation family: $\nu = \frac{d^2}{ds^2}\Big|_{s=0} \mathrm{Scal}_{\mathrm{vert}}(\omega, J_s)$

Introduced by Dervan-Sektnan when the fibres are cscK.

Here: extension to K-semistable fibres

§ OPTIMAL SYMPLECTIC CONNECTIONS

$$\boxed{p_E(\Delta_{vert}\Lambda_{\omega_B}F_{\mathcal{H}} + \Lambda_{\omega_B}\rho_{\mathcal{H}} + \lambda\nu) = 0}$$

- LHS is smooth function. p_E projection onto $\Gamma^\infty(E \to B) =: \mathcal{C}^\infty(E)$

E_b = holomorphy potentials on X_b = holomorphic vector fields on X_b that vanish somewhere $= \{f \in \mathcal{C}^\infty(X_b) \mid \bar\partial \nabla^{1,0} f = 0\}$

Our assumption from before that $\mathrm{Aut}_0(X_b, T^{1,0}X_b)$ are all isomorphic implies that their Lie algebras $\mathfrak{h}_0(b)$ have all the same dimension and $\mathfrak{h}_0(b) \longleftrightarrow E_b$

$\Rightarrow$ $E \to B$ is a vector bundle [Hallam]

Why extend optimal symplectic connections to K-semistable fibres?

Because it is an open condition while cscK it is not!

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Rmk: the equation is interesting when the fibres have more automorphisms of the total space. Eg. it is trivial when the fibres are Rieman surfaces

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E.g. projective bundles:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)^\vee \arrow[d] & \mathcal{E} \arrow[d] \\ \mathbb{P}(\mathcal{E}) \arrow[r] & B \end{tikzcd}$$

h Hermitian metric on

$\rightsquigarrow$ $h^\vee$ Hermitian metric on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)^\vee$

Its curvature $\omega = iF_{h^\vee}$ such that $\omega|_{\mathbb{P}(\mathcal{E})_b} = \omega_{FS}|_b$

ω is optimal symplectic connection $\iff$ A_h is Hermite Einstein

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Assume: $\mathrm{Aut}(\pi_Y) = \{f \in \mathrm{Aut}(Y, H_Y) \mid \pi \circ f = \pi\}$ discrete

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Going back to the Kuranishi space $\bar{V}_\pi$:

$$0 \in \bar{V}_\pi \longleftrightarrow \begin{array}{c} X \\ \downarrow \\ B \end{array}$$

Let $v_0 = \partial_s|_{s=0}\, y_s \in T_0 \bar{V}_\pi$. Then we can write the equation on Y as: $\boxed{\Theta(\omega, 0, v_0) = 0}$

$$y_s \in V_\pi \longleftrightarrow \begin{array}{c} X_s \\ \downarrow \\ B \end{array} \simeq \begin{array}{c} Y \\ \downarrow \\ B \end{array}$$

§ OPTIMAL SYMPLECTIC CONNECTIONS

$$\boxed{\Theta(\omega, 0, v_0) = 0}$$

Assume • $Y \to B$ has an optimal symplectic connection, i.e. $\Theta(\omega, 0, v_0) = 0$.

• $W \subset V_\pi$ locally closed subset of the first lemma: $\forall w \in W$ $Y_w \to B$ admits a degeneration to $X' \to B$ rel. cscK.

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Lemma (–) Let $\mathrm{Aut}(\pi_Y)$ be discrete. Let $w \in W \longleftrightarrow Y_w \to B$.

Then we can find a pair $(x, v) \in TW$ s.t.

$x \longleftrightarrow X' \to B$ cscK fibres

$v = \partial_s|_{s=0} w_s \in T_x V_\pi$

Then $\exists \tilde{\omega} \in c_1(H_Y)$ s.t. $\Theta(\tilde{\omega}, x, v) = 0$

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Then $\exists\, \tilde{\omega} \in c_1(H_Y)$ s.t. $\Theta(\tilde{\omega}, x, v) = 0$

Pf: implicit function theorem

Rmk: the Lemma gives openness of solutions within a locally closed subvariety.

MODULI SPACE

Let $Y \to B$ admit an optimal symplectic connection

The two Lemmas give a locally closed complex space W where the equation still admit solutions

$\Rightarrow$ Local charts of moduli space : $\quad W / \mathrm{Aut}(\pi_Y) \quad$ where $\mathrm{Aut}(\pi_Y)$ is finite.

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μ-stability $\longleftrightarrow$ $\exists$ HE
K-stability $\longleftrightarrow$ $\exists$ cscK

- Hausdorff
- Has a Weil-Petersson type kähler metric

Remark :

- $\exists$ optimal symplectic connections $\overset{?}{\longleftrightarrow}$ fibration stability [Dervan-Sektnan] [Hallam]
  f-stability [Hattori]

- [Hashizume-Hattori] moduli space of Calabi-Yau fibrations over a curve where also the base changes

Thank you