Modularity of Landau–Ginzburg models

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Mirror Symmetry for Fano varieties

Approaches to Mirror Symmetry for Fano varieties

Idea

Mirror Symmetry corresponds to a Fano variety X its *Landau–Ginzburg model* — a quasi-projective variety Y equipped with a regular function $\mathrm{w}\colon Y \to \mathbb{C}$ such that its fibres are mirror dual to anticanonical sections of the Fano variety X.

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Homological MS (Kontsevich, Auroux, Katzarkov, Seidel. . .)

For a Fano variety X and a smooth anticanonical divisor V the log Calabi–Yau pair (X, V) has a mirror log Calabi–Yau pair (Z, D). Compactification of $X \setminus V$ to X corresponds to equipping $Y = Z \setminus D$ with a proper function $\mathrm{w}\colon Y \to \mathbb{C}$.

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Toric MS (Batyrev–Kim–Ciocan-Fontanine–van Straten, Eguchi–Hori–Xiong...)

If X is a Fano variety, and X admits a degeneration to a Fano toric variety T, then the mirror of X admits a torus chart $(\mathbb{C}^*)^n$ to which $\mathrm{w}\colon Y \to \mathbb{C}$ restricts to give a Laurent polynomial p whose Newton polytope is the anticanonical polytope of T.

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- the family $\mathsf{p}\colon (\mathbb{C}^*)^{\dim(X)} \to \mathbb{C}$ admits a fibrewise compactification $\mathsf{w}\colon Y \to \mathbb{C}$ such that Y is a smooth non-compact Calabi–Yau variety;

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- there should exist a degeneration of X to a toric variety T whose fan polytope is the Newton polytope of p.

Convention

We will often identify the Laurent polynomial p with the family $\mathsf{p}\colon (\mathbb{C}^*)^{\dim(X)} \to \mathbb{C}$ or its compactification $\mathsf{w}\colon Y \to \mathbb{C}$.

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Fanosearch project (Fanosearch team)

An inverse process: by characterizing the Laurent polynomials that appear in this way, one should be able to *create* a family of Fano manifolds starting from the data of a Laurent polynomial of appropriate type by applying deformation-theoretic techniques.

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The main challenge has been to characterize which polynomials correspond to Fano varieties. The current expectation is that there is a bijection between toric Gorenstein Fano varieties and mutation classes of *rigid maximally mutable Laurent polynomial*.

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For a smooth Fano threefold X and its general anticanonical section V the restriction map $\mathrm{Pic}(X) \xrightarrow{\mathrm{res}} \mathrm{Pic}(V)$ is an isomorphism. The deformation space of pairs (X, V) forms a complete family of $\mathrm{Pic}(X)$-polarized K3 surfaces.

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Theorem (Ilten–Lewis–Przyjalkowski, Fanosearch team)

Toric Landau–Ginzburg models exist for smooth Fano threefolds.

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Claim

There exists a mirror-dual statement to Beauville's theorem for toric Landau–Ginzburg models of smooth Fano threefolds.

Mirror Symmetry of lattice-polarized K3 surfaces

Lattice polarizations of K3 surfaces

Notation

We refer to the lattice $L_{\mathrm{K3}} = H^{\oplus 3} \oplus \mathbb{E}_8(-1)^{\oplus 2}$ as the *K3 lattice*.

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Definition

Let L be an even lattice of signature $(1, r)$. We say that a K3 surface S is *L-polarized* if there is a primitive embedding $\iota \colon L \hookrightarrow \mathrm{Pic}(S)$ whose image contains an ample class.

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A family $\varpi \colon \mathcal{S} \to B$ of K3 surfaces is *L-polarized* if there is a trivial sub-local system $\mathbb{L} \subseteq R^2\varpi_*\underline{\mathbb{Z}}_{\mathcal{S}}$ which induces an L-polarization on each fibre of the family ϖ.

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Theorem (Dolgachev)

Let L be a lattice such that there exists a unique (up to isometry) primitive embedding into the K3 lattice L_{K3}. Then there exists a coarse moduli space $\mathcal{M}_L$ of L-polarized K3 surfaces.

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Remark

There exist sufficient conditions on a lattice L to have a primitive embedding into the K3 lattice, and to ensure that this embedding is unique. More generally, the coarse moduli space of L-polarized K3 surfaces has finite number of irreducible components corresponding to equivalence classes of embeddings of L into L_{K3}.

Dolgachev–Nikulin duality

Definition

Let $L \subset L_{\mathrm{K3}}$ be a primitive sublattice. Assume that the orthogonal complement admits the decomposition $L^{\perp} = H \oplus L^{\vee}$ for some lattice $L^{\vee}$. We refer to $L^{\vee}$ as the *Dolgachev–Nikulin dual* to L.

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Dolgachev–Nikulin duality is a form of Mirror Symmetry for lattice-polarized K3 surfaces: for a primitive sublattice $L \subset L_{\mathrm{K3}}$ it interchanges complete families of L and $L^\vee$-polarized K3 surfaces.

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Theorem (Dolgachev)

Let $\mathcal{F}$ and $\mathcal{F}^{\vee}$ be complete families of L- and $L^{\vee}$-polarized K3 surfaces. We have $\dim(\mathcal{F}) = \rho(S)$, where $S \in \mathcal{F}^{\vee}$ is general.

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Corollary

Let X be a smooth Fano threefold. Then $\dim(\mathcal{M}_{\mathrm{Pic}(X)^{\vee}}) = \rho(X)$.

Dolgachev–Nikulin duality for smooth Fano threefolds

Definition

A compactification of the family $\mathsf{p}\colon (\mathbb{C}^*)^n \to \mathbb{C}$ to a family $\mathsf{f}\colon Z \to \mathbb{P}^1$, where Z is smooth, and $-K_Z \sim \mathsf{f}^{-1}(\infty)$, is called a *log Calabi–Yau compactification*.

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Theorem (Przyjalkowski)

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Conjecture

Let X be a smooth Fano threefold, and F be a general fibre of its toric Landau–Ginzburg model $\mathsf{f}\colon Z \to \mathbb{P}^1$. There exists an isometry

$$\mathrm{Pic}(X)^\vee \cong \mathrm{im}\left(H^2(Z,\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(F,\mathbb{Z})\right).$$

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Conjecture holds for Fano threefolds with $\rho(X) = 1$.

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Definition (see Cheltsov–Przyjalkowski, 2018)

Let X be a smooth Fano threefold with $\rho(X) > 1$. We refer to the toric Landau–Ginzburg model of the Fano threefold X used in *loc. cit.* as a *standard* Landau–Ginzburg mirror of X.

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Remark (see Akhtar–Coates–Galkin–Kasprzyk)

If $-K_X$ is very ample, then a standard LG model is given by a *Minkowski polynomial*. They are unique up to explicitly described birational transformations of $(\mathbb{C}^*)^{\mathrm{dim}(X)}$ called *mutations*.

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Theorem (DHKOP)

Conjecture holds for standard toric Landau–Ginzburg models. In each case the dual lattice $\mathrm{Pic}(X)^\vee$ was explicitly constructed.

Tame compactifications and deformation theory of Landau–Ginzburg models

Tame compactifications

Definition (Katzarkov–Kontsevich–Pantev)

A *proper, tame compactified Landau–Ginzburg model* is a triple (Z, D, f) consisting of a smooth projective variety Z, a simple normal crossings divisor D, and a morphism $\mathrm{f}\colon Z \to \mathbb{P}^1$ so that $\mathrm{f}^*(\infty) = D$, where D is an anticanonical divisor of Z.

We put $Y = Z \setminus D$ and denote by w the restriction of f to Y.

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Remark

The notion of log Calabi–Yau compactification differs from the notion of tame compactification by not requiring snc condition for the fiber over infinity. However, in most of known cases of log Calabi–Yau compactifications, in particular, for standard toric Landau–Ginzburg models, the fibers over infinity are snc. These notions may become different if we allow singularities over infinity.

Tame compactifications

Theorem (Katzarkov–Kontsevich–Pantev (1), DHKOP (2))

Suppose (Z, D, f) is a tame compactified Landau–Ginzburg model of dimension 3 so that (Z, D) is a log Calabi–Yau pair satisfying certain minor topological conditions. Let F be a smooth fibre of f.

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2. The forgetful map $\mathrm{Def}(Z, F) \to \mathrm{Def}(F)$ is a submersion of relative dimension $h^{2,1}(Z)$ onto the subspace of $\mathrm{Def}(F)$ preserving the polarization by monodromy invariants L_Z.

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Consequently, for a general deformation of (Z, D, f) and general fibre F of f, the Picard lattice of F is isomorphic to L_Z.

Specialization to the case of Laurent polynomials

Let us recall that a toric Landau–Ginzburg model for a smooth Fano threefold with with very ample anticanonical class is given by a *Minkowski polynomial*: a three-dimensional Laurent polynomial with reflexive Newton polytope and special integral coefficients.

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Theorem (DHKOP)

Let X be a smooth Fano threefold with very ample anticanonical class, and let (Z, D, f) be its standard Landau–Ginzburg model, obtained by partially compactifying the corresponding Laurent polynomial p of X. There is a $\rho(X)$-dimensional family of Landau–Ginzburg models deformation equivalent to (Z, D, f) s.t.

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1. Any small deformation of Z is obtained by deforming p.
2. The deformation space of pairs (Z, F) form a complete family of $\mathrm{Pic}(X)^\vee$-polarized K3 surfaces.

Geometry of moduli spaces

Let $\overline{\mathcal{M}}_L$ be the *Baily–Borel compactification* of $\mathcal{M}_L$. It is obtained by adding points called *type III boundary components* and curves called *type II boundary components* to $\mathcal{M}_L$.

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The type III (respectively, type II) boundary components are in set-theoretic bijection between $O^+(L^\perp)$ equivalence classes of rank 1 (respectively, rank 2) totally isotropic sublattices of $L^\perp$.

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Remark

The existence of a type III boundary point in $\mathcal{M}_L$ implies that the lattice $L^\perp$ admits a totally isotropic sublattice of rank 1. In fact, Dolgachev–Nikulin mirror symmetry suggests more, namely, that there is an embedding of the lattice H into $L^\perp$.

Definition

Suppose $\mathbb{V}$ is a $\mathbb{Q}$-local system over a nonempty Zariski-open subset of $\mathbb{P}^1$. Let $i \colon U \to \mathbb{P}^1$ denote the canonical embedding. We say that $\mathbb{V}$ is *extremal* if $H^1(\mathbb{P}^1, i_*\mathbb{V}) = 0$.

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Given a map $\phi\colon \mathbb{P}^1 \to \overline{\mathcal{M}}_{L_Z}$, there is a finite number of variations of Hodge structure over $\phi^{-1}(\mathcal{M}_{L_Z})$ for which the period map is ϕ. We say that the map ϕ is *extremal* if one of these variations of Hodge structure has extremal underlying local system.

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Corollary (DHKOP)

Suppose X is a Fano threefold with very ample anticanonical bundle. Then the moduli space of $\mathrm{Pic}(X)^\vee$-polarized K3 surfaces admits a type III boundary point p_∞ and a ruling by extremal curves passing through p_∞.

Thank you!