On divisorial stability of finite covers

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1. Motivation
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1. Motivation

Stability theories aim to have two goals:

1) construct moduli spaces
2) answer questions in differential geometry.

K-stability: developped in 90's - 2000s to answer which Fano varieties admit a Kähler-Einstein metric (Yau-Tian-Donaldson (YTD) conjecture).
Fano varieties: $-K_X$ ample (or equivalently the Ricci curvature) Very important in algebraic geometry due to the MMP.
K-stability is heavily modeled on GIT.

Original definition via test configurations: Consider a $(X;L)$ (Fano) variety X with ample line bundle L (for Fano $L=-K_X$).

Def: A test configuration for $(X;L)$ is a variety $\mathcal{X}$ with
(i) A $\mathbb{Q}$-line bundle $\mathcal{L}$; $(\mathcal{X};\mathcal{L})$
(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting on $\mathcal{L}$.
(iii) a flat $\mathbb{G}_m$-equivariant morphism $\pi:\mathcal{X}\to\mathbb{A}^1$
such that each fibre $(\mathcal{X}_t;\mathcal{L}_t)$ for $t\neq 0$ is isomorphic to $(X;L)$

The Futaki invariant is an analytic invariant measuring obstruction to having a KE metric (for Fanos).

for $k\in\mathbb{N}$ $d_k := \dim H^0(X,kL) = a_0k^n + a_1k^{n-1} + O(k^{n-2})$, $a_i\in\mathbb{Q}$.
For central fibre $(\mathcal{X}_0;\mathcal{L}_0)$, $\mathbb{C}^*$ action $H^0(\mathcal{X}_0,k\mathcal{L}_0)$
with weight w_k given by the equivariant Riemann-Roch by $w_k = b_0k^{n+1} + b_1k^n + O(k^{n+1})$

Def The generalised Futaki invariant of $(\mathcal{X};\mathcal{L})$ is

$$\mathrm{Fut}(\mathcal{X},\mathcal{L}) := \frac{a_1b_0 - a_0b_1}{a_0^2}$$

Def: $(X;L)$ is 1) K-semistable if for any t.c $(\mathcal{X};\mathcal{L})$ of $(X;L)$
$\mathrm{Fut}(\mathcal{X},\mathcal{L})\geq 0$
2) K-polystable if for any t.c. $(\mathcal{X};\mathcal{L})$ of $(X;L)$
$\mathrm{Fut}(\mathcal{X},\mathcal{L})\geq 0$ with equality if $(\mathcal{X};\mathcal{L})$ trivial
3) K-stable if for any t.c. $(\mathcal{X};\mathcal{L})$ of $(X;L)$

$\mathrm{Fut}(x, y) \geq 0$ with equality if $x \cong X \times \mathbb{A}^1$.

Theorem (Chen-Donaldson-Sun) If X is Fano it admits a KE metric iff it is K-poly stable $(X; -K_X)$

This is the link of K-stability of Fanos with differential geometry.

In recent years K-stability of Fanos is studied through birational geometry.

Def Let X be a proj. variety $\pi: Y \longrightarrow X$, Y variety, birational morphism a prime divisor $F \subset Y$ is called a prime divisor over X denoted F/X.

Let $A_X(F) = 1 + \mathrm{ord}_F(K_Y - \pi^* K_X)$ the log discrepancy.

$$S_{(X;L)}(F) = \frac{1}{\mathrm{vol}(L)} \int_0^\infty \mathrm{vol}(\pi^* K_X - tF)\, dt.$$

$$\beta_{X;L}(F) = A_X(F) - S_X(F)$$

where $\mathrm{vol}(F) = \limsup_{k \to \infty} \frac{h^0(X, \mathcal{O}_X(kL))}{k^n/n!}$. if F is big

$\mathrm{vol}(F) > 0$. If F is nef $\mathrm{vol}(F) = F^n$.

Theorem (Fujita-Li) For X Fano, $(X; -K_X)$ is

(i) K-semistable iff $\beta_X(E) \geq 0$ for all E/X.
(ii) K-stable iff $\beta_X(E) > 0$ for all E/X.

The use of this definition + a plethora of techniques from birational geometry has allowed to show the following:

Theorem (Codogni, Blum, Jiang, Li, Liu, Patakfalvi, Xu, Zhuang, ...)

There exists a moduli stack $\mathcal{M}^K$ admitting a projective good moduli space M^K parametrising K-semistable Fano varieties (with $L = -K_X$).

This answers goal 1. A natural question to ask is whether for arbitrary polarized variety $(X; L)$ K-stability can show a similar theorem. (goal 1). For goal 2 the generalized YTD conjecture predicts that K-polystable $\Leftrightarrow$ existence of cscK metrics on X.

2. Divisorial stability.

For general $(X; L)$ the β invariant used above can't determine K-stability. Also working with t.c. makes it almost

impossible to prove these properties. We need a new approach.
The approach used here uses nonArchimedean geometry (NA). (Boucksom - Jonsson 22

<u>NA Preliminaries</u>: Consider $\mathbb{C}$, with the trivial norm.

For X projective, we define the <u>Berkovich Analytification</u> of X, denoted by X^{an}, as the space of all semivaluations of X i.e. valuations $v : k^{\times}(Y) \longrightarrow \mathbb{R}$, for $Y \subseteq X$ a subvariety.

$\underline{X^{div}}$ is the space of divisorial valuations $v = c \operatorname{ord}_F$ for F/X.

$[C^0(X)]$ is the space of cts functions $\phi : X^{an} \to \mathbb{R}$ with dual space $\underbrace{C^0(X)^\vee}$. (Radon probability measures).

We can define a function on X^{an} associated to a t.c as follows:

Assume $\underline{(\mathcal{X}, \mathcal{L})}$ dominates the trivial t.c. i.e. $\exists$ map $\underbrace{\mathcal{X} \xrightarrow{\rho} X \times \mathbb{A}^1}$, and write $\underline{\mathcal{L} - L = D}$ (L the pullback of L along ρ;

ρ: $\mathcal{X} \supseteq \mathcal{X}_0 = \sum_j b_j E_j$ and define $\underbrace{\phi_{(\mathcal{X},\mathcal{L})}}$ via the relationship:

$$\underbrace{\phi_{(\mathcal{X},\mathcal{L})}(a \operatorname{ord}_{E_j}) = b_j^{-1} a \operatorname{ord}_{E_j}(D).}$$

$$\mathcal{X} \xrightarrow{\rho} X \times \mathbb{A}^1 \xrightarrow{pr} X.$$
$$\overset{\cap}{E_j}.$$

<u>Def</u>: A <u>Fubini-Study metric</u> is a metric $\underline{\phi_{(\mathcal{X},\mathcal{L})}}$ associated to a nef t.c. $(\mathcal{X}, \mathcal{L})$. A <u>psh metric</u> is a uniform limit of a sequence of FS metrics on X^{an}. The set of FS is $\mathcal{H}^{NA}(X)$

<u>Def</u>: The <u>Monge-Ampère energy</u> of a FS metric is
$$E(\phi_{(\mathcal{X},\mathcal{L})}) = E(\mathcal{X}, \mathcal{L}) := \frac{\mathcal{L}^{n+1}}{(n+1) L^n}.$$

This extends to psh metrics by setting $\overbrace{E(\phi) = \inf \{ E(\phi_{(\mathcal{X},\mathcal{L})}) \mid \phi_{(\mathcal{X},\mathcal{L})} \geq \phi \}}$.
Here $\phi_{(\mathcal{X},\mathcal{L})} \geq \phi_{(\mathcal{X}',\mathcal{L}')}$ if $\exists$ $\mathcal{Y}$ with $\mathbb{G}_m$-equivariant morphisms to $\mathcal{X}, \mathcal{X}'$ s.t. that $\underline{\mathcal{L} - \mathcal{L}'}$ $(\text{on } \mathcal{Y})$ is effective. We also define $\mathcal{E}^1 := \{ \phi \text{ psh} \mid E(\phi) \text{ finite} \}$.

$$\mathcal{X} \longleftarrow \mathcal{Y} \longrightarrow \mathcal{X}' \quad \left((\mathcal{X}, \mathcal{L}),\ (\mathcal{Y}, \mathcal{L}) \quad \begin{matrix} \mathcal{Y} \\ \downarrow \\ \mathcal{X} \end{matrix} \right).$$

For a measure $\underline{\mu \in C^0(X)^\vee}$ we define its <u>norm</u>
$$\underline{\|\mu\|_L} := \sup_{\phi \in \mathcal{E}^1} \{ \underbrace{E(\phi)} - \int_{X^{an}} \phi \mu \} \in [0, \infty].$$

$$\underbrace{\mathcal{M}^1 = \{ \mu \mid \|\mu\|_L < \infty \}}$$

For an $\mathbb{R}$-divisor H we denote

$$\underbrace{\nabla_H \|\mu\|_L} := \frac{d}{dt}\Big|_{t=0} \|\mu\|_{L+tH}.$$

Denote by $\underbrace{\delta_{c\,\mathrm{ord}_F}}$ the Dirac mass supported on $\underbrace{v = c\,\mathrm{ord}_F}$ such that $\left\{\int \phi\, \delta_v = 1 \quad \text{if} \quad \phi(v) \neq 0\right\}$.

Example if $H = L = -K_X$ and X Fano then for $\underbrace{\mu = \delta_{\mathrm{ord}_F}}$:

$$\nabla_H \|\mu\|_L = \underbrace{-S_X(F)}.$$

Def Divisorial measures are probability measures of the form

$\mu = \underbrace{\sum_j a_j \delta_{v_j}}$ for a finite collection of divisorial valuations $\underline{v_j}$ with $\sum_j a_j = 1$. $\underline{\mathcal{M}^{div}}$ is the set of divisorial measures.

The entropy $\mathrm{Ent}_X : \mathcal{M}^{div} \to \mathbb{R}$ is

$\mathrm{Ent}_X(\mu) = \underbrace{\int_{X^{an}} A_X \mu}$. If μ is as above, $v_j = c_j \mathrm{ord}_{F_j}$,

$$\mathrm{Ent}_X(\mu) = \sum_j a_j A_X(v_j) = \sum_j a_j c_j A_X(F_j)$$

The β invariant is then defined as

$$\beta_{(X;L)}(\mu) := \underbrace{\mathrm{Ent}_X(\mu)} + \underbrace{\nabla_{K_X} \|\mu\|_L}.$$

X is Fano $L = -K_X$, $\mu = \delta_v$
$\beta_{(X,-K_X)}(\mu) = \beta_X(F)$.

Def (Boucksom – Jonsson 22') $(X;L)$ is

(i) divisorially semistable if for all divisorial measures μ on X^{an} $\beta(\mu) \geq 0$

(ii) divisorially stable if $\exists \varepsilon > 0$ st for all divisorial measures μ on X^{an} $\beta(\mu) \geq \varepsilon \|\mu\|_L$

Theorem (Boucksom-Jonsson) $(X;L)$ divisorially stable $\Rightarrow$ $(X;L)$ uniquely cscK $\Rightarrow$ $(X;L)$ uniformly K-stable.

Conjecture (YTD) $(X;L)$ uniformly K-stable $\Rightarrow$ $(X;L)$ uniquely cscK $\Rightarrow$ $(X;L)$ divisorially stable

3. Finite covers. (with R. Dervan)

First, we introduce the notion of equivariant divisorial stability.

Let $G \curvearrowright X$ then $G \curvearrowright X^{div}$ by $(g \cdot v)(f) = v(g^* f)$.

Def A divisorial measure $\mu = \sum a_j \delta_{v_j}$ is G-invariant if for all $g \in G$,

$$\mu = \sum_j a_j \delta_{g(v_j)}$$

$\mathcal{M}^{div,G}_X$ is the space of G-invariant divisorial measures.

Def $((X,B);L)$ is

(i) G-equivariantly divisorially semistable if for all $\mu \in \mathcal{M}^{div,G}$, $\beta(\mu) \geq 0$

(ii) G-equivariantly divisorially stable if for all $\mu \in \mathcal{M}^{div,G}$ $\exists \varepsilon$ $\beta(\mu) \geq \varepsilon \|\mu\|_L$.

Set-up: let (X, L_X), (Y, L_Y) be normal projective varieties such that

$\pi : (Y, L_Y) \to (X, L_X)$ is a Galois cover

s.t. $G \curvearrowright Y$ with $X := Y/G$. Let Δ_X, Δ_Y effective $\mathbb{Q}$-divisors such that $K_Y + \Delta_Y = \pi^*(K_X + \Delta_X)$ then:

(Dervan-P. 23')

Theorem $((Y, \Delta_Y); L_Y)$ is G-equivariantly divisorially (semi-) stable if and only if $((X, \Delta_X); L_X)$ is divisorially (semi-) stable.

Idea of proof: define pullbacks and pushforwards of divisorial measures and metrics on X^{an} and Y^{an}

i.e. for $\nu \in \mathcal{M}^{div}_X$ define $\pi^*\nu \in \mathcal{M}^{div}_Y$

$\mu \in \mathcal{M}^{div}_Y$ define $\pi_*\mu \in \mathcal{M}^{div}_X$.

$\phi \in \mathcal{E}^1_X$ define $\pi^*\phi \in \mathcal{E}^1_Y$

$\phi \in \mathcal{E}^1_Y$ define $\pi_*\phi \in \mathcal{E}^1_X$.

Show that $\mathcal{M}^{div,G}_Y \cong \mathcal{M}^{div}_X$ and $\mathcal{E}^{1,G}_Y \cong \mathcal{E}^1_X$. ($\pi^*\pi_*\mu = \mu$, if $\mu \in \mathcal{M}^{d}$)

Show that $\beta(\pi^*\pi_*\mu) = \beta(\mu)$. □

Theorem (Dervan-P 23'). Let (X, L_X) be a divisorially

[illegible] (X, L_X) [illegible]
semistable smooth polarized variety. There is a $k>0$ such that: if we let

(i) $B \in |kL|$ be such that (X,B) is log canonical
(ii) $\pi : Y \to X$ be the m-fold cover of X branched over B
(iii) assume Y is smooth

then Y admits cscK metric in $c_1(L_Y)$ where $L_Y = \pi^* L_X$.

Idea of proof:

1) show that there exists k s.t. for any $B \in |kL|$ such that (X,B) is log canonical, then $((X,B),L)$ is log divisorially stable.

2) show that if (X,L_X) is divisorially semistable and $((X,B);L)$ is log divisorially stable then $((X,cB);L)$ is divisorially stable for all $0 < c \leq 1$.

3) combine 1+2 + the first theorem to show (Y, L_Y) is G-equivariantly divisorially stable

4) show by a BJ-type argument that G-equivariantly divisorially stable $\Rightarrow$ cscK metric. if smooth. $(X, ((1-\frac{1}{m})B); L_X)$ [illegible] (Y, L_Y) div. stable

Remark: The existence of the metrics in this Theorem is also a consequence of work of Arezzo-Della Vedova-Shi which doesn't use divisorial stability but completely analytic techniques. The above proof is completely algebraic.