0. X smooth Fano / $\mathbb{C}$ (i.e. X proj, $-K_X$-ample)

When is X K-(poly) stable?

$\Updownarrow$ YTD $\quad \longleftarrow$ algebraic condition on degenerations of X

$\exists$ a Kähler-Einstein metric on X

How to tell if X is K-stable?

Tian (1987): α-invariant $\quad n = \dim X$

If $\alpha(X) > \frac{n}{n+1} \implies X$ is K-stable.

I. Complex α-invariant.

KLT-singularities: (X, Δ) pair ($K_X + \Delta$ is $\mathbb{Q}$-Cartier)

normal variety $\quad \geq 0$ $\mathbb{Q}$-divisor

- defined using log-resolutions of (X, Δ)
- KLT: roughly - log-resolution is "close" to (X, Δ).

$$\mathrm{lct}(X, \Delta) = \sup\{t \geq 0 \mid (X, t\Delta) \text{ is KLT}\}$$

log-canonical threshold

Tian, Demailly: X KLT Fano ($(X,0)$ is KLT).

$$\alpha(X) = \sup\left\{ t \geq 0 \;\middle|\; \begin{array}{l} (X, tD) \text{ is KLT } \forall\ D \geq 0\ \mathbb{Q}\text{-div} \\ \text{s.t. } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}$$

$$= \inf_{n \geq 1} \inf_{D \sim -nK_X} \left(\operatorname{lct}(X, D) \cdot n \right)$$

Fact: $\alpha(X) > 0$ (not clear if X is not smooth).

E.g. $X = \mathbb{P}^n$ $\quad -K_X = (n+1) H \leftarrow$ hyperplane

$$\alpha(X) = \frac{1}{n+1} = \operatorname{lct}\left(X, (n+1)H\right)$$

II Global F-regularity: k - perfect field, char $p>0$

(X,Δ) pair$/k$ $\qquad X \xrightarrow{F^e} X$ (locally $r \mapsto r^{p^e}$)

Frobenius map $\qquad$ ring homomorphism

Def: (X,Δ) is globally F-regular (GFR) if for any effective divisor $D \geq 0$, $\exists$ some $e \gg 0$ s.t. the map

(Weil)

$$\begin{tikzcd} \varphi: \mathcal{O}_X \arrow[r] & F^e_*\mathcal{O}_X \arrow[r] & F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D) \arrow[ll, bend right, dashed, "\psi"'] \end{tikzcd}$$

Frobenius map

splits $/\mathcal{O}_X$ i.e. $\exists\ \psi$ s.t. $\psi \circ \varphi = id$

$\mathcal{O}_X$-mod map

Fact (Schwede-Smith): X Fano$/\mathbb{C}$ $\quad D \sim_{\mathbb{Q}} -K_X$

Then For $0 \le t < 1$

(X, tD) is KLT $\iff$ (X, tD) mod p is GFR $\forall\, p \gg 0$.

III. α_F-invariant.

X globally F-regular Fano$/k$

$$\alpha_F(X) = \sup\left\{ t \ge 0 \;\middle|\; \begin{array}{l} (X, tD) \text{ is globally F-regular} \\ \forall\, D \ge 0 \ \mathbb{Q}\text{-div} \quad D \sim_{\mathbb{Q}} -K_X \end{array} \right\}$$

Remarks: 1) Let $S = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mrK_X))$ s.t. $-rK_X$ - Cartier

$Y = \operatorname{Spec}(S) \leftarrow$ cone over $(X, -rK_X)$.

More precise analogy (X, tD) is GFR $\overset{\text{Analogy}}{\approx}$ $(Y, t\tilde{D})$ is KLT

2) $\alpha_F(X) = \inf_{n \geq 1} \inf_{D \sim -nK_X} \left(\operatorname{fpt}(Y, \tilde{D}) \cdot n \right)$ $\leftarrow$ F-pure threshold. — char $p>0$ analog of lct.

3) If $0 \leq \Delta \sim_{\mathbb{Q}} -K_X$ $\operatorname{lct}(Y, \tilde{\Delta}) = \min\{1, \operatorname{lct}(X, \Delta)\}$

$\therefore$ $\alpha_F(X) \overset{\text{Analogy}}{\approx} \min\{1, \alpha(X)\}$

IV. Thm (_) : X globally F-regular Fano / k

Then 1) $\alpha_F(X) > 0$ (also follows from results of Kenta Sato)

2) $\alpha_F(X) \leq 1/2$

3) $\alpha_F(X) = 1/2 \iff s(X) = \dfrac{\mathrm{vol}(-K_X)}{2^d (d+1)!}$ $d = \dim X$

F-signature of X

4) If X is a toric Fano variety, then

$\alpha_F(X) = \alpha_{\mathbb{C}}(X)$ ← complex α-invariant

Rem: (4) + (2) $\Rightarrow$ $\alpha_{\mathbb{C}}(X) \leq 1/2$ for toric Fano X.

Examples: 1) $X = Q_d = \{ X_0^2 + X_1^2 + \dots + X_{d+1}^2 = 0 \} \subseteq \mathbb{P}^{d+1}$

$$\alpha_F(X) = \alpha_{\mathbb{C}}(X) = \frac{1}{d}.$$

2) $X = \{ x^3 + y^3 + z^3 + w^3 = 0 \} \subseteq \mathbb{P}^3$

$\alpha_p = \alpha_F(X/\mathbb{F}_p)$. Then 1) $\alpha_p > 0$ for $p \geq 5$

2) $\alpha_p < 1/2$ for $\forall p \geq 5$

3) $\lim_{p \to \infty} \alpha_p = 1/2$.

Uses work of Shideler-Tucker, Han-Monsky

$$\alpha_{\mathbb{C}}(X) = 2/3$$

V. Reason for Part (2). i.e. $\alpha_F(X) \leq 1/2$

Duality for Frobenius:

$$\mathcal{H}om_{\mathcal{O}_X}(F_*^e \omega_X^{-m}, \mathcal{O}_X) \cong F_*^e \omega_X^{-(p^e-1-m)} \quad \forall m.$$

$\rightsquigarrow$ this gives a symmetry in the Frobenius splittings.

VI. Semicontinuity:

Thm 2: [-] $f: \mathcal{X} \to Y$ flat family of GFR Fanos

Assume Y is smooth $/k$

i.e. $-K_{\mathcal{X}/Y}$ is $\mathbb{Q}$-Cartier, and f-ample.

Then, the map $Y \ni y \longmapsto \alpha_F(\mathcal{X}_{y^\infty})$

is lower-semicontinuous.

Rmk: Corresponding result for $\alpha_{\mathbb{C}}$-invariant is due to Blum-Liu - using Nadel vanishing & global generation results.