Integral affine geometry & degenerations of Calabi-Yau manifolds (joint w. E. Mazzon)

Let $X \xrightarrow{\pi} \mathbb{D}^*$ projective degeneration of CY mflds
(X smooth, π subm, proper)

- $X_t = \pi^{-1}(t)$ sm. CY $K_{X_t} \sim \mathcal{O}_{X_t}$
- meromorphic at $t=0$: $X \subset \mathbb{P}^N \times \mathbb{D}$
  eq. of X_t merom/t

maximally degenerate case:

X_t breaks into as many pieces as possible as $t \to 0$

<u>example:</u>

$$X = \{ x_0 \cdots x_{n+1} + t f_{n+2} = 0 \} \subset \mathbb{P}^{n+1} \times \Delta^*$$

SYZ conjecture: $\exists$ $f_t : X_t \to B$

special Lagrangian torus fib. $\quad$ real dim n

smooth over $B^0 \subseteq B$

$\Delta = B \setminus B^0$ $\quad$ codim $\Delta \geq 2$

Action-angle coordinates: $\exists$ $\nabla^{\mathbb{Z}}$ on B^0

$\mathbb{Z}$-aff structure, atlas with tr. functions in $GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$

<u>Goal</u> reconstruct (B, B°, ∇^Z)

using non-archimedean geometry

(following Kontsevich - Soibelman)

+ NA SYZ fibration

I) Models & dual complexes

Def $\underset{\text{sm.}}{\mathfrak{X}} \xrightarrow{\pi} \mathbb{D}$ (snc) model of X if:

- $\mathfrak{X}_{\mathbb{D}^\times} \simeq X$
- $\mathfrak{X}_0 = \sum_{i \in I} n_i \underset{\text{sm.}}{E_i}$ $\qquad E_I = \bigcap E_j$

Rq snc models always exist: Hironaka

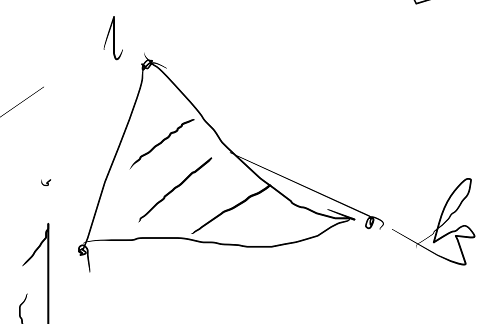
$Y \subseteq \mathfrak{X}_0$ stratum $\iff \exists\, J \subseteq I$,

$Y =$ c.c. of $\bigcap_{j \in J} E_j$

<u>Dual complex</u>: $E_i \subseteq \mathfrak{X}_0 \rightsquigarrow v_i$

c.c. of $E_i \cap E_j \rightsquigarrow e_{ij}$ edge $v_i \to v_j$

$E_i \cap E_j \cap E_k$



$Y \subseteq \mathfrak{X}_0$ stratum $\Leftrightarrow$ $Y =$ c.c. of $\bigcap_{j \in J} E_j$

faces of $\mathcal{D}(\mathfrak{X}) \leftrightarrow$ strata of $\mathfrak{X}_0$

σ_Y face of $\sigma_{Y'}$ $\Leftrightarrow$ $Y' \subseteq Y$

Def $X \xrightarrow{\pi} \mathbb{D}^*$ max. degenerate

iff $\forall\, \mathcal{X}$, $\dim \mathcal{D}(\mathcal{X}) = \dim X_t$

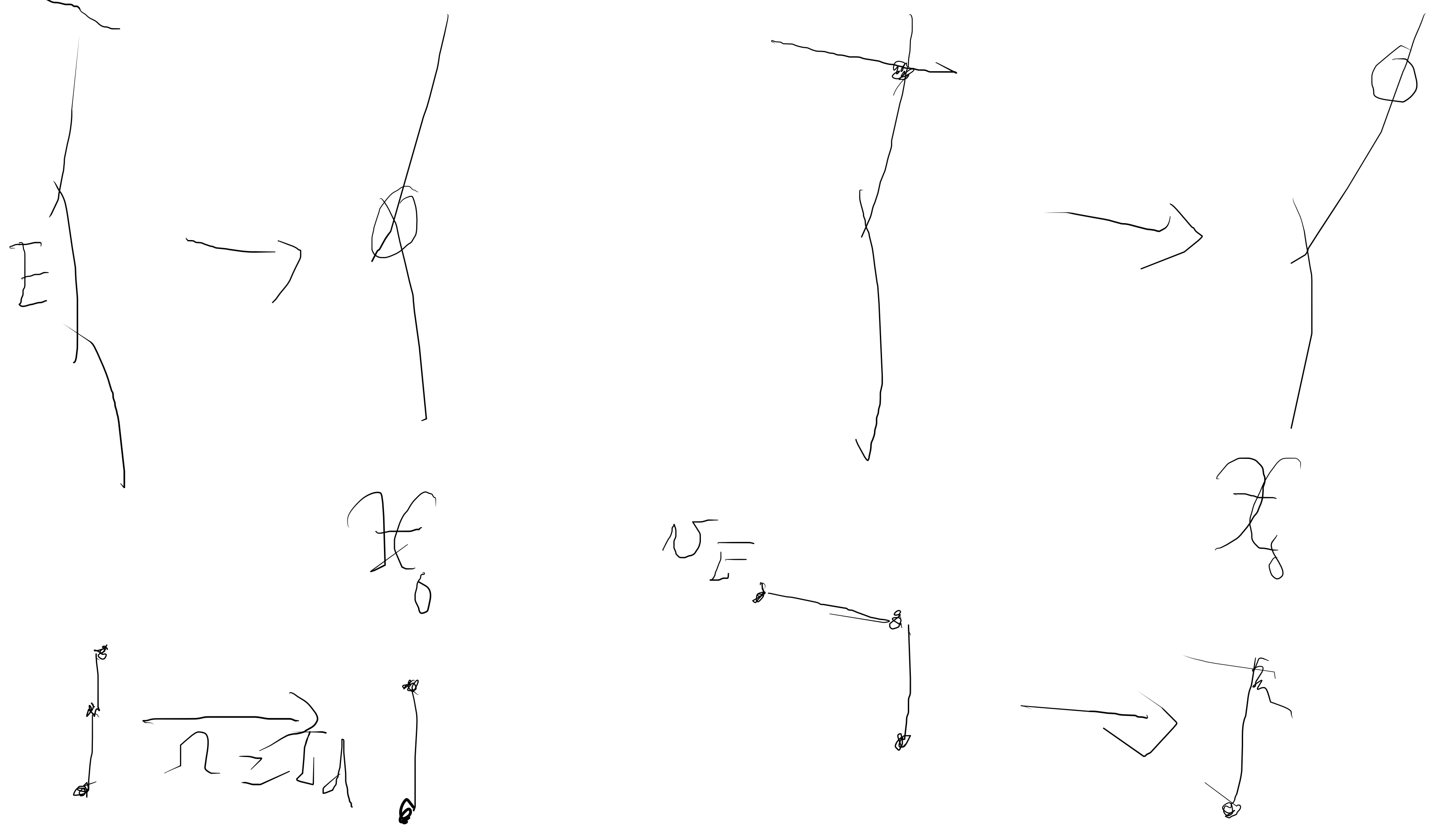
Prop $\mathcal{X}' \longrightarrow \mathcal{X}$ morphism of models;

$\exists\, r_{\mathcal{X}'\mathcal{X}} : \mathcal{D}(\mathcal{X}') \longrightarrow \mathcal{D}(\mathcal{X})$ PA

i section of r

embedding.

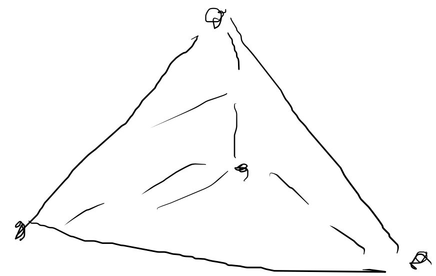
examples



<u>example</u> $\mathcal{X} = \{x_1 \cdots x_3 + t f_4 = 0\} \subseteq \mathbb{P}^3 \times \mathbb{D}$

$$\mathcal{X}_0 = x_0 x_1 x_2 x_3 = 0$$

$D(\mathcal{X}) =$



$\simeq \partial \Delta^3$

⚠ $\mathcal{X}$ not snc

$$\mathrm{Sing}(\mathcal{X}) = \bigcup_{i \neq j} \{t = x_i = x_j = f_4 = 0\}$$

E_i, E_j are not Cartier at $p \in \mathrm{Sing}$

e.g. $\mathcal{X}' \to \mathcal{X}$ $\quad \exists\, r_{\mathcal{X}'/\mathcal{X}} : D(\mathcal{X}') \to D(\mathcal{X})$

Insight of Kontsevich–Soibelman:

$\mathcal{B} \subseteq \mathcal{D}(\mathfrak{X}) \quad \forall \mathfrak{X}$:

$\exists \quad \mathcal{D}(\mathfrak{X})^{ess} \subseteq \mathcal{D}(\mathfrak{X})$

ind of $\mathfrak{X}$, intrinsic to X

Thm (Nicaise-Xu-Yu '16)

Let $\mathcal{X} \to D$ minimal model of X in the sense of MMP

i.e. $K_{\mathcal{X}/D} + \left((\mathcal{X}_0)_{red}\right) \sim \mathcal{O}_{\mathcal{X}}$

then: $\mathcal{D}(\mathcal{X}) \simeq \mathcal{D}(\mathcal{X})^{ess}$

smallest you can get

Rq
- $\mathcal{X}$ mildly singular
  $D(\mathcal{X})$ still ok
- $\mathcal{X}$ exists when X algebraic
  but not unique
- $D(\mathcal{X})$ unique as a set but
  the triangulation may change

II) NA geometry

$X \subseteq \mathbb{P}^N \times \mathbb{D}^* \rightsquigarrow X \subseteq \mathbb{P}^N_K$

$K = \mathbb{C}((t))$ NA field $\quad f = \sum_{n \geq n_0} a_n t^n$

$|f|_K = e^{-\mathrm{ord}_0(f)} \qquad \mathrm{ord}_0(f) = \min\{n \mid a_n \neq 0\}$

$X/K \rightsquigarrow$ Berkovich analytic space X^{an}

X^{an} nice topological space:

$\begin{cases} \text{Hausdorff} \\ \text{loc. compact} \\ \text{loc. contractible} \end{cases}$

compact $\Leftrightarrow$ X/K proper

X^{an} = space of valuations on $K(X)$

$X^{an} \supseteq \{ v_x : K(X) \to \mathbb{R} \ / \ v_{|K} = \text{ord} \}$

$v_x = -\log |\cdot|_x$ ← NA abs. value

Prop $\mathfrak{X}/D$ snc model : $i_{\mathfrak{X}} : \mathcal{D}(\mathfrak{X}) \hookrightarrow X^{an}$

image $:= Sk(\mathfrak{X}) \subseteq X^{an}$

Moreover, $X^{an} \xrightarrow[\text{homeo}]{\sim} \varprojlim_{\mathfrak{X} \text{ snc models}} Sk(\mathfrak{X})$

$\rightsquigarrow$ induces $\rho_{\mathfrak{X}} : X^{an} \longrightarrow Sk(\mathfrak{X})$

"NA SYZ fibration"

v_i vertex of $\Delta(\mathcal{X})$

$E_i \subseteq \mathcal{X}_0$ $\qquad$ $\ell_{\mathcal{X}}(v_i) = \mathrm{ord}_{D_i}$

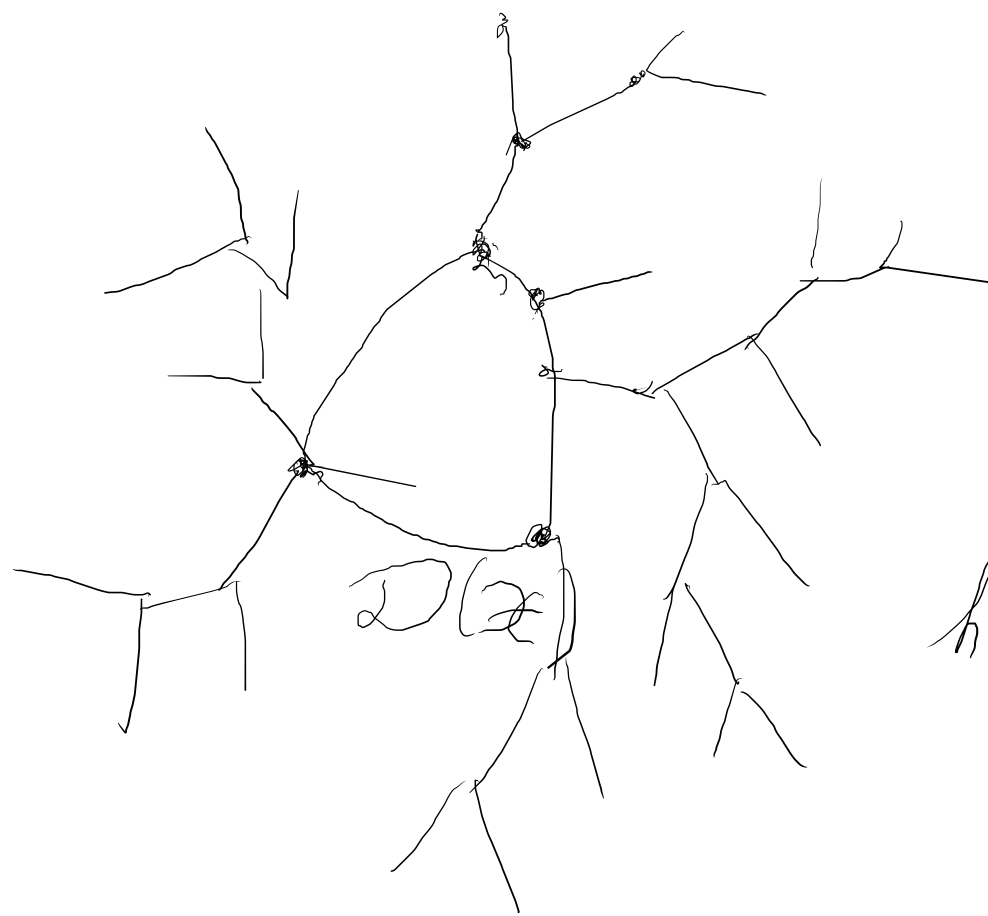
then we can interpolate

to embed $\Delta_{ij} \hookrightarrow X^{an}$

example $\mathcal{X} \subseteq \mathbb{P}^2_R \quad R = \mathbb{C}[[t]]$

$$\mathcal{X} = \{x_0 x_1 x_2 + t f_3 = 0\}$$

$X^{an} =$



homotopic $\widetilde{\ } \; S^1$

$X \to \mathbb{D}$ Calabi-Yau, $Sk(X) := Sk(\mathcal{X})$ for $\mathcal{X}$ minimal $\subseteq X^{an}$

NA analog of smooth torus fibration?

$\underline{\rho : X^{an}} \longrightarrow Sk(X)$

$T = \operatorname{Spec} K[T_1^{\pm}, \dots, T_n^{\pm}]$

$T^{an} \xrightarrow{\text{val}} \mathbb{R}^n$

$x \longmapsto (-\log |T_1|_x, \dots, -\log |T_n|_x)$

$$(\mathbb{C}^\times)^n \longrightarrow \mathbb{R}^n$$

$$(x_1, \ldots, x_n) \longmapsto (-\log|x_1|, \ldots, -\log|x_n|)$$

Def B top space, $\rho : X^{an} \longrightarrow B$ continuous

ρ affinoid torus fibration at $x \in B$

iff: $\exists\, U \ni x \subseteq B$

$$\begin{tikzcd}
\rho^{-1}(U) \arrow[r, "\sim", "\text{iso. of an spaces}"'] \arrow[d, "\rho"'] & \mathrm{val}^{-1}(V) \subseteq T^{an} \arrow[d, "\mathrm{val}"] \\
U \arrow[r, "\text{homeo}"'] & V \subseteq \mathbb{R}^n
\end{tikzcd}$$

induces a ∇^Z on U = pullback of $\nabla^Z_{\mathbb{R}^n}$

Thm (Nicaise-Xu-Yu '18

Mazzon-PS '21)

$\mathcal{X} \to \mathbb{D}$ dlt deg of proj. varieties

s.t. the components of $\mathcal{X}_0$ $\mathbb{Q}$-Cartier

(snc ok)

$Z \subseteq \mathcal{X}_0$ stratum s.t. $\begin{cases} (Z, \Delta_Z) \text{ toric} \\ N^*_{Z/\mathcal{X}} \text{ nef} \end{cases}$

$\Delta_Z = \sum_{E_i \not\supseteq Z} E_i \cap Z$

Δ_Z toric boundary

$D_i \cap Z$ connected $\forall D_i \subseteq \mathcal{X}_0$

then $e_{\underline{z}}$ affinoid torus fibration

over $\mathrm{Star}(\sigma_{\underline{z}}) = \bigcup\limits_{\substack{\sigma \supseteq \sigma_{\underline{z}} \\ \text{faces of } D(\underline{z})}} \sigma$

+ explicit descr. of ∇^{Z} on $\mathrm{Star}(\sigma_{\underline{z}})$

in terms of $Z_{\underline{z}}$ fam of Z

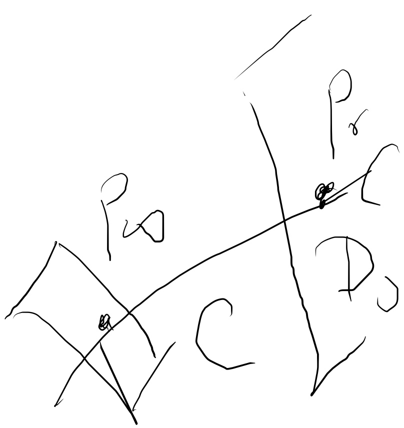
examples

- $Z = \{pt\}$ $\qquad \mathrm{Star}(\sigma_Z) = \mathring{\sigma}_Z$

n-simplex

P_Z a pt.

- $Z = (\mathbb{P}^1, [0] + [\infty])$



if $\mathcal{X}$ min-model

$\forall\, C \subseteq \mathcal{X}_0$ strata,

$C \simeq \mathbb{P}^1$

$\Rightarrow \ell_{\mathcal{X}} : X^{an} \longrightarrow \underline{Sk(X)}$

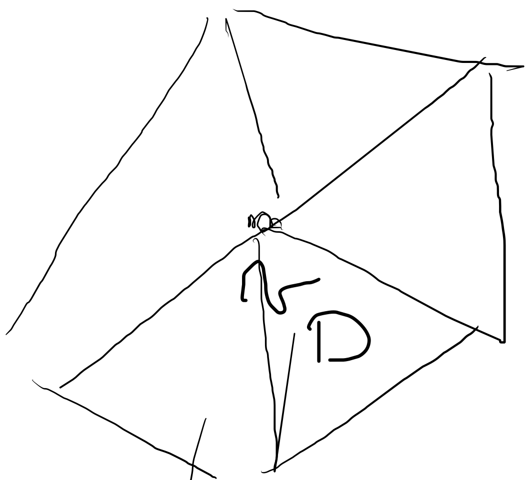
a t f around each codim 1 face

$\Delta = (n-2)$-skeleton

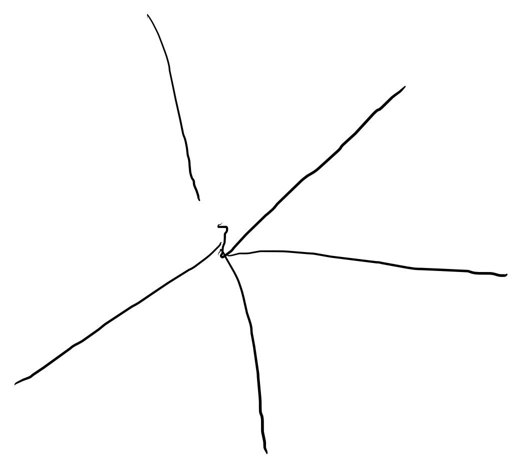
then $B^0 = B \setminus \Delta$ and $\ell_{\mathcal{X}}$ sm/ore B^0

• $Z = D$ irr. comp of $\mathcal{X}_0$

$(N_D, \vee Z)$



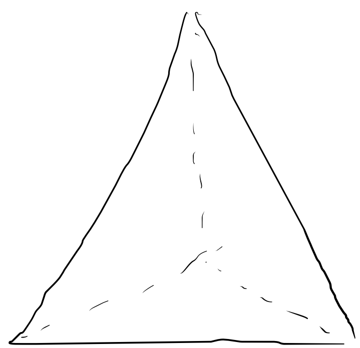
$\mathrm{Star}(N_D) \setminus L \simeq (0, \Sigma_D)$



$\Sigma_D \subseteq \mathbb{R}^2$

III) Hypersurfaces in $\mathbb{P}^3$

$$\mathcal{X} = \{x_0 \cdots x_3 + t f_4 = 0\} \subset \mathbb{P}^3 \times \mathbb{D}$$

$\mathcal{D}(\mathcal{X}) =$  $= \mathrm{Sk}(X)$

sing. at $x_i = x_j = t = f_4 = 0$ $\qquad \cancel{w} = -f_4$

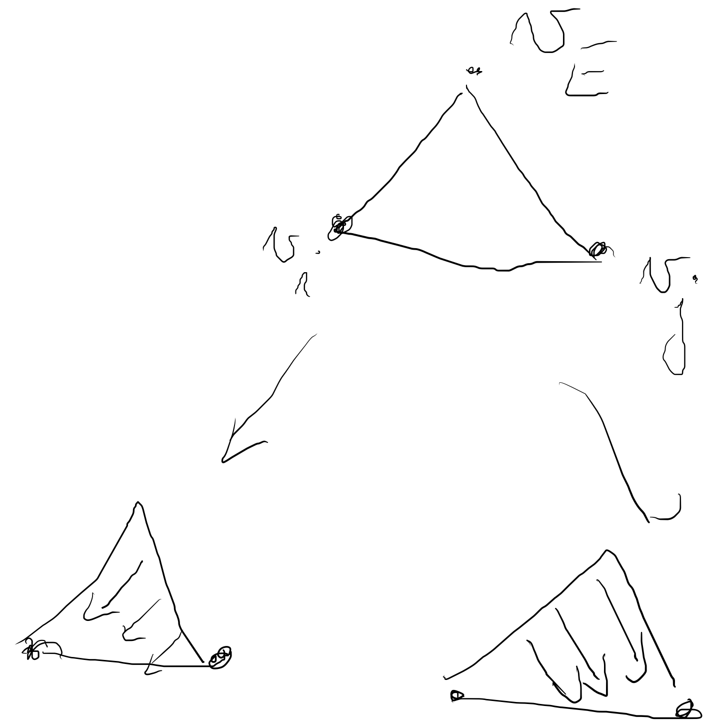
$$\mathcal{U} = \{x_i x_j = t w\} \subset \mathbb{A}^3 \times \mathbb{D}$$

$D_i = \{x_i = t = 0\}$ not $\mathbb{Q}$-Cartier

$\Rightarrow$ no $\rho_{\underline{X}}$

naive resolution: $B_0\mathcal{U} \longrightarrow \mathcal{U}$

$E \subset \mathbb{P}^1 \times \mathbb{P}^1$



$$\begin{tikzcd} & B_0\mathcal{U} \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ \mathcal{U}_i \arrow[dr] & & \mathcal{U}_j \arrow[dl] \\ & \mathcal{U} & \end{tikzcd}$$

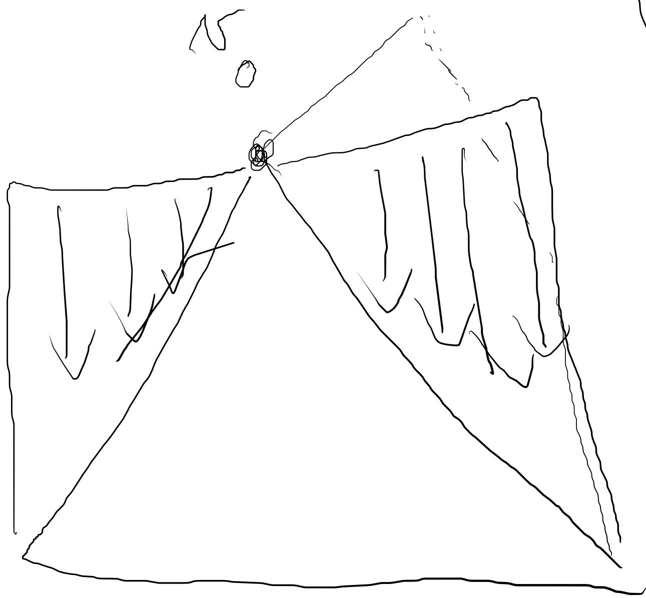
preserves toricness of $D_j = \{x_j = 0\}$

st. transform $\tilde{D}_j \subset \mathcal{U}_j$, $\tilde{D}_j \xrightarrow{\simeq} D_j$

$D_0' \simeq D_0$ $\qquad$ $D_0 = (\mathbb{P}^2, \ell_1 + \ell_2 + \ell_3)$

$\mathscr{C}_{\tilde{\mathfrak{X}}}$ $\qquad$ $\tilde{\mathfrak{X}} = B_{24}(\mathfrak{X})$

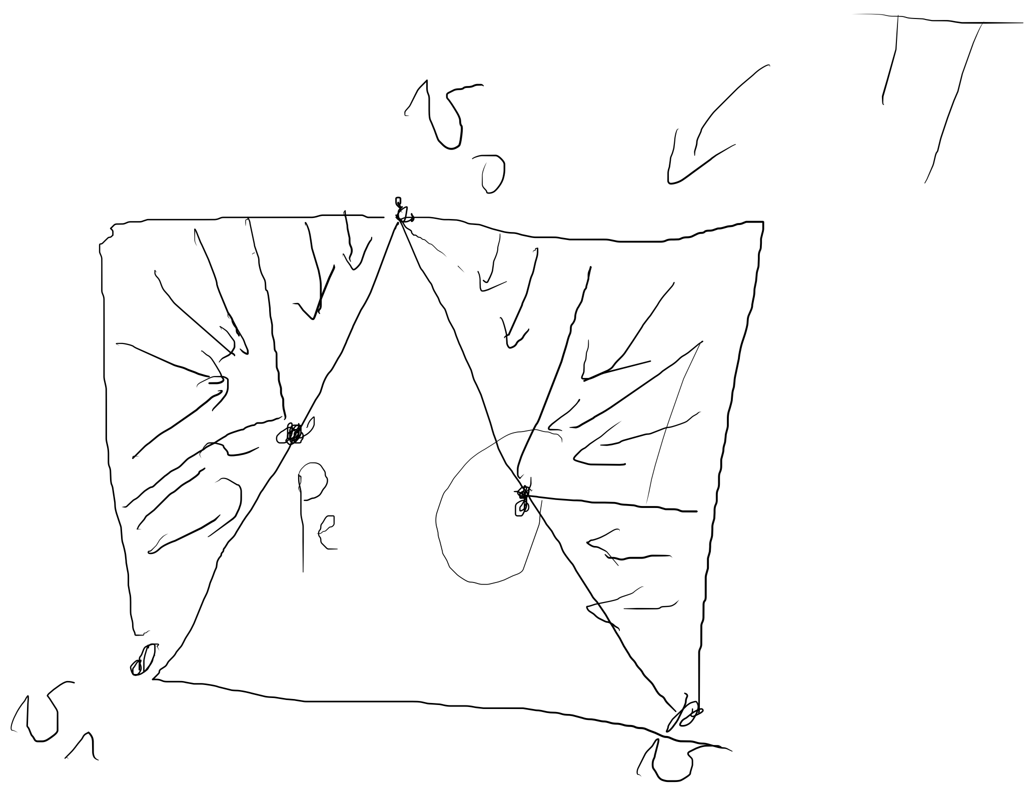
v_0



$\hookrightarrow D(\tilde{\mathfrak{X}})$

then $\mathscr{C}_{\tilde{\mathfrak{X}}_0}$ is an affinoid torus fibration at v_0

C_{S_4}



then $\rho = \pi \circ \rho_{\tilde{x}}$ $\cdot$ at g at each vertex

$\tilde{x} = B_{24}$

Thm (Mazzon, PS '21)

There exists a similar construction for generic quintic threefolds

we obtain $\mathbb{Z}$- affine structures occurring in the Gross - Siebert program