Categorified Beauville-Laszlo & related problems

joint with Federico Binda (arXiv: 2107.03914 v2)

0. Introduction & motivations

X = a scheme

$Br(X) := H^2_{\acute{e}t}(X;\mathbb{G}_m)$ the cohomological Brauer group of X

It is a fundamental arithmetic and algebro-geometric invariant of X:

- it measures obstructions to Hasse principles for existence of rational points;
- it is a birational invariant ⇝ examples of unirational varieties that are not rational

Deal with $Br(X)$? Fundamental idea (Grothendieck): look at the table

n	0	1	2
Algebro-geometric meaning of $H^n_{\acute{e}t}(X;\mathbb{G}_m)$	Invertible functions	Line bundles = invertible sheaves $H^1_{\acute{e}t}(X;\mathbb{G}_m) = Pic(X)$	derived Azumaya algebras $\shortparallel$ invertible sheaves of categories B. Toën (2012)

Goal of this work: exploit Toën's viewpoint to improve our knowledge of $Br(X)$.

2 main applications:

⟶ to formal GAGA situation

⟶ to Beauville-Laszlo situation ⇝ openings to a larger research program

Plan of the talk:

1. Review of derived Azumaya algebras
2. Review of categorical sheaves
3. Formal GAGA setup
4. Beauville-Laszlo setup

1. Azumaya algebras & their derived counterparts

Def. X a scheme.

1) A sheaf of Azumaya algebras is a pair $\mathcal{A} = (V, m)$, where

(i) V is a vector bundle on X

(ii) $m: V \otimes_{\mathcal{O}_X} V \longrightarrow V$ is an associative multiplication not commutative

Moreover, we require that the canonical map

$$V \otimes V^{op} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(V, V) \qquad (b, b') \longmapsto [c \longmapsto \underbrace{bcb'}_{\text{Azum. multipl.}}]$$

to be an isomorphism

2) Two sheaves of Azumaya algebras $\mathcal{A}_1, \mathcal{A}_2$ are said to be Morita equivalent if there exists an $\mathcal{O}_X$-linear equivalence $\mathcal{A}_1\text{-Mod} \simeq \mathcal{A}_2\text{-Mod}$

Examples 1) $\mathcal{O}_X$ is Azumaya

2) Fix $a, b \in \mathbb{Z}$ and let p be a prime s.t. $(a,b)_p = 1$ (i.e. $z^2 = ax^2 + by^2$ has non-trivial solution mod p)

Over $X = \operatorname{Spec}(\mathbb{Z}_{(p)})$,

$$A_{a,b} = \mathbb{Z}_{(p)}\langle i, j, k\rangle \big/ (i^2 - a, j^2 - b, ij - k, ji + k)$$

is an Azumaya algebra.

3) $\mathcal{A}$ is Morita equivalent to $\mathcal{O}_X \iff \mathcal{A} \simeq M_{n\times n}(\mathcal{O}_X) \underset{\text{isom as alg.}}{\simeq} \operatorname{End}(\mathcal{O}_X^n)$

$\mathrm{Mod}_{M_{n\times n}(\mathcal{O}_X)} \simeq \mathrm{Mod}_{\mathcal{O}_X}$

Notation: $\mathrm{Br}_{Az}(X) := \{\text{Azumaya algebras on } X\} / \text{Morita equiv.}$

Tensor product makes $\mathrm{Br}_{Az}(X)$ into a group. $A \otimes B$ is still Azumaya if A & B are

An extra example:

4) The s.e.s. of étale sheaves

$$0 \longrightarrow \mathbb{G}_m \longrightarrow GL_{n+1} \longrightarrow PGL_n \longrightarrow 0$$

gives rise to $H^1_{et}(X; PGL_n) \xrightarrow{\delta} H^2_{et}(X; \mathbb{G}_m)$.

In fact, it factors through $\mathrm{Br}_{Az}(X)$.

Facts. 1) Étale locally, every Azumaya algebra is Morita trivial "Azumaya alg. are twisted forms of matrix algebras"

2) $\exists$ injective map $Br_{Az}(X) \xhookrightarrow{i} Br(X)$

3) The image of i is contained in $Br(X)_{tors}$

4) (Mumford) $\exists$ normal surface S st. $Br(S)/Br(S)_{tors} \neq 0$

$\boxed{\rightsquigarrow \exists \text{ classes in the Brauer group not representable by Azumaya algebras!}}$

Rmk. $H^1_{et}(X; \mathbb{G}_m) \simeq Pic(X)$ is great because we know how to manipulate line bundles!

Thm (Toën)

Every class in $Br(X)$ is representable by a derived Azumaya algebra.

$V_\bullet \quad 0 \quad \cdots V_2 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow V_{-1} \longrightarrow V_{-2} \cdots \quad 0$

Def. X a scheme.

1) A sheaf of derived Azumaya algebras is an object $\mathcal{A} \in Alg(\mathcal{P}erf(X))$ such that:

$V_\bullet \otimes V_\bullet \longrightarrow V_\bullet$ associative mult. in complexes of v.b.

(i) $\mathcal{A}$ is supported everywhere;

E.g. $X = \mathbb{A}^1$

$V = (k[T] \xrightarrow{T} k[T]) \in \mathcal{P}erf(\mathbb{A}^1)$

$H^i(V)$ supp. just on 0

(ii) the canonical map

$$\mathcal{A} \otimes \mathcal{A}^{op} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A})$$

$k[T] \xrightarrow{\cdot T} k[T]$ complex

is a quasi-isomorphism.

2) Two sheaves of derived Azumaya algebras $\mathcal{A}_1, \mathcal{A}_2$ are derived equivalent if

$\exists$ a $\mathcal{O}_X$-linear equivalence $\mathcal{A}_1\text{-dgMod} \simeq \mathcal{A}_2\text{-dgMod}$

$Ch(\mathcal{O}_X) =$ chain complexes in $QCoh(X)$

$\mathcal{A}_1 \in Alg(Ch(\mathcal{O}_X))$ $\quad Mod_{\mathcal{A}_1}(Ch(\mathcal{O}_X))[W^{-1}_{q.is.}]$ $\simeq$ $\mathcal{A}_1$-dgMod

Remark. Toën's thm + Mumford example $\Longrightarrow$ $\boxed{\text{existence of derived Azumaya algebras that are not classical}}$

However, it is somehow difficult to construct explicit examples.

It's easier to produce derived Azumaya algebras from the categorified viewpoint.

Cor. (Toën) $\exists$ a can. surj. map

$dBr(X) \twoheadrightarrow Br(X)$

$\wr\wr$ Thm

$H^2_{et}(X; \mathbb{G}_m) \oplus \underbrace{H^1_{et}(X; \mathbb{Z})}_{} \simeq 0$ if X is normal

Notation: $dBr(X) := \left\{ \begin{array}{c} \text{derived Azumaya} \\ \text{algebras} \end{array} \right\} \Big/ \text{derived Morita}$

2. Categorical sheaves

Notation & conventions

- All categories are ∞-categories (think of triang. categories but better!)
- $\mathcal{P}r^{L,\omega}$ = presentable ∞-categories generated by compact objects
  Morphisms: left adjoints that preserve compact objects

Present. = describable via generators & relation.

Ex. X top. space
$PSh(X) \hookleftarrow$ a free category
$\downarrow$
$Sh(X) \leftarrow$ a quotient of a free cat.

- $\mathcal{C}, \mathcal{D} \in \mathcal{P}r^L$. $\mathcal{C} \otimes \mathcal{D}$ corepresents

$F: \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{E}$
$\forall c$ $F(c,-)$ comm. with colim
$\forall d$ $F(-,d)$ " " "

Lurie

$$Fun^L(\mathcal{C}\otimes\mathcal{D}, \mathcal{E}) \overset{def.}{\simeq} Fun^{L\times L}(\mathcal{C}\times\mathcal{D}, \mathcal{E})$$

$$\mathcal{C}\otimes\mathcal{D} \overset{Thm}{\simeq} Fun^R(\mathcal{C}^{op}, \mathcal{D})$$

Mod_A generated by A

Examples to forge intuition.

1) $PSh(\mathcal{C}_0)\otimes PSh(\mathcal{D}_0) \simeq PSh(\mathcal{C}_0\times\mathcal{D}_0)$

$\mathcal{C}\otimes\mathcal{C} \xrightarrow{m} \mathcal{C}$

2) $Sh(X)\otimes Mod_A \simeq Sh(X; Mod_A)$

$\mathcal{C}\times\mathcal{C}$ $\otimes$ tensor structure for $\mathcal{C}$

3) $\left\{\text{Commutative algebras in } \mathcal{P}r^L\right\} = \left\{\begin{array}{c}\text{Symmetric monoidal } \infty\text{-categories whose tensor product} \\ \text{commutes with colims in both variables}\end{array}\right\}$

4) X a qcqs scheme. $D(X) \in CAlg(\mathcal{P}r^{L,\omega})$
categorifies $\mathcal{O}_X$

$3\frac{1}{2}$) Mod_A cat. pres. $\otimes_A : Mod_A \times Mod_A \longrightarrow Mod_A$
bicontinous.

Def. X a qcqs scheme. Categorical quasi-coherent sheaves $:= \mathcal{P}r_X^{L,\omega} := Mod_{D(X)}(\mathcal{P}r^{L,\omega})$

$D(X)\otimes\mathcal{C} \longrightarrow \mathcal{C}$ module structure

Remarks & examples

1) Action of $D(X) \equiv$ action of $\mathcal{O}_X$ (categorification)

$D(X)\otimes D(Y) \longrightarrow D(Y)$
$\mathcal{F} \quad \mathcal{G} \longmapsto Lf^*(\mathcal{F})\otimes_{\mathcal{O}_Y}\mathcal{G}$

2) $f: Y \longrightarrow X$. Then $D(X) \overset{Lf^*}{\curvearrowright} D(Y)$, $D(Y)\in\mathcal{P}r_X^L$.
cat. of $Rf_*\mathcal{O}_Y$

3) The above definition is "cheating": unclear why these are sheaves.

They are though (thms of Toën & Lurie), and the above definition is equivalent

to a more sophisticated one (thm of Gaitsgory).

Thm (Toën)

$Pic(X) \hookrightarrow QCoh(X)$ $\otimes$-inv. sheaves

X qcqs scheme. Write:

$dAz_X^{cat} \hookrightarrow \mathcal{P}r_X^{L,\omega}$

- dAz_X^{cat} = full subcategory of $\mathcal{P}r_X^{L,\omega}$ spanned by $\otimes$-invertible objects
- dAz_X = maximal ∞-groupoid in dAz_X^{cat}.

Then dAz_X is a simplicial set and moreover:

$(\mathcal{L}, [n])$ shift

$\pi_0(dAz_X) \simeq dBr(X) \simeq H^2_{ét}(X;\mathbb{G}_m) \times H^1_{ét}(X;\mathbb{Z})$

$\pi_1(dAz_X) \simeq dPic(X) \simeq H^1_{ét}(X;\mathbb{G}_m) \times H^0_{ét}(X;\mathbb{Z})$

$\pi_2(dAz_X) \simeq H^0(X;\mathbb{G}_m)$

$\pi_i(dAz_X) \simeq 0 \quad \forall i \geq 3$

---

3. Formal GAGA setup

Setup:

- $S = \mathrm{Spec}(A)$, $(A, \mathfrak{m})$ complete local ring $\quad \mathbb{C}[[T]]$
- $X \longrightarrow S$ a proper scheme $/S$
- $S_n = \mathrm{Spec}(A/\mathfrak{m}^{n+1})$, $X_n = S_n \times_S X$
- $\mathfrak{X} = \mathrm{colim}\, X_n$ formal completion of X at the special fiber

$Pic(\mathfrak{X}) \simeq \lim Pic(X_n)$

Fact: Grothendieck's existence thm $\Longrightarrow$ $H^1_{ét}(\mathfrak{X};\mathbb{G}_m) \simeq \lim_n H^1_{ét}(X_n;\mathbb{G}_m)$

Question: what about $H^2_{ét}(\mathfrak{X};\mathbb{G}_m)$?

Thm (Grothendieck)

Assume that:

1) A is a DVR;

2) X is regular and flat in addition to proper $/S$;

3) $\lim^1_n Pic(X_n) = 0$.

Then $H^2_{ét}(X;\mathbb{G}_m) \longrightarrow \lim_n H^2_{ét}(X;\mathbb{G}_m)$ is injective.

$$0 \to \underset{\substack{\text{\rotatebox{90}{$\simeq$}}\\ Pic}}{\lim} \to \prod_n Pic(X_n) \to \prod_n Pic(X_n) \to \lim^1 Pic(X_n) \to 0$$

$$(\mathcal{L}_n) \longmapsto (\mathcal{L}_n - \mathcal{L}_{n+1}|_{X_n})$$

Def. $H^2_{ét}(\mathfrak{X};\mathbb{G}_m) := H^2(\lim_n R\Gamma_{ét}(X_n;\mathbb{G}_m))$

Thm (Binda - P.)

Assume that:

1) A complete local ring;

2) X proper $/S$

Then :

1) the map $H^2_{ét}(X;\mathbb{G}_m) \longrightarrow H^2_{ét}(\mathfrak{X};\mathbb{G}_m)$ is injective;

2) there exists a s.e.s.:

$H^2_{ét}(X;\mathbb{G}_m)$ target map

$$0 \to \lim_n{}^1 \mathrm{Pic}(X_n) \longrightarrow H^2_{ét}(\mathfrak{X};\mathbb{G}_m) \longrightarrow \lim_n H^2_{ét}(X_n;\mathbb{G}_m) \longrightarrow 0$$

Recovers and strengthens all previously known results. It follows from:

> **Thm** (Binda - P.)
>
> In the above setting the map
>
> $$\mathcal{P}r^{L,\omega}_X \longrightarrow \lim_n \mathcal{P}r^{L,\omega}_{X_n}$$
>
> is fully faithful on dualizable objects $\supseteq$ invertible object = Azumaya.

Corollary (Binda - P.)

In the above setting, let $\mathcal{A} \to X$ be a $\mathbb{G}_m$-gerbe. $\mathcal{A} \simeq X \times B\mathbb{G}_m$ étale locally

not proper

Then the canonical map

$$\mathrm{Perf}(\mathcal{A}) \longrightarrow \lim_n \mathrm{Perf}(\mathcal{A}_n)$$

is an equivalence.

This also goes beyond existing literature: Combining results of Rydh, Hall, Totaro, Vistoli et al., this was only known for $\mathbb{G}_m$-gerbes that are global quotients!

4. Beauville-Laszlo setup

Setup (simplified)

A: Noetherian commutative ring

$I \subset A$ ideal

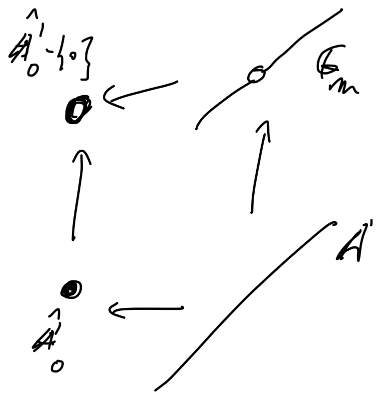
$$\begin{tikzcd}
V = T \setminus V(I\hat{A}_I) \arrow[r, "g"] \arrow[d, "j"'] & U = S \setminus V(I) \arrow[d, "i"] \\
T = \mathrm{Spec}(\hat{A}_I) \arrow[r, "f"] & S = \mathrm{Spec}(A)
\end{tikzcd}$$

Significant example:

$A = \mathbb{C}[T]$

$I = (T)$

$$\begin{tikzcd}
\mathbb{C}((T)) & \mathbb{C}[T, T^{-1}] \arrow[l] \\
\mathbb{C}[[T]] \arrow[u] & \mathbb{C}[T] \arrow[l] \arrow[u]
\end{tikzcd}$$



Thm (Beauville-Laszlo, Lurie)

The canonical map

$$D(S) \longrightarrow D(T) \times_{D(V)} D(U)$$

is an equivalence of ∞-categories.

$$\mathrm{Vect}(\mathbb{C}[T]) \simeq \mathrm{Vect}(\mathbb{C}[[T]]) \times_{\mathrm{Vect}(\mathbb{C}((T)))} \mathrm{Vect}(\mathbb{C}[T,T^{-1}])$$

$\longleftarrow$ It plays a really important role in geometric Langlands

Thm (Bindo - P.)

The canonical map

$$\mathcal{P}_{2\,S}^{L,w} \longrightarrow \mathcal{P}_{2\,T}^{L,w} \times_{\mathcal{P}_{2\,V}^{L,w}} \mathcal{P}_{2\,U}^{L,w}$$

$\otimes$ symm. monoidal $\otimes$

is an equivalence of ∞-categories

symm. mon.

Pass to $\rightsquigarrow$ inv. objects

Rmk. Full faith. is easy

Ess. surj is somehow surprising

$$\mathrm{Pic}(\mathcal{P}_{2\,S}^{L,w}, \otimes) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{P}_{2\,T}^{L,w}, \otimes) \times_{\mathrm{Pic}(\mathcal{P}_{2\,V}^{L,w}, \otimes)} \mathrm{Pic}(\mathcal{P}_{2\,U}^{L,w})$$

$\wr\wr$ Thm. Toën

dAz_X $\longleftarrow$ What does it mean?

X qcqs scheme.

$\mathcal{C} \in \mathcal{P}_{2\,X}^{L,w} = \mathrm{Mod}_{D(X)}(\mathcal{P}_2^{L,w})$

Thm (Toën) If $\mathcal{C}$ is invertible then $\exists\, E \in \mathcal{C}$ such that

$$\mathcal{C} \simeq \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(E)}(D(X))$$

$+ \mathrm{End}_{\mathcal{C}}(E) \in D(X)$ is Azumaya

Thm (Bondal-Orlov)

X qcqs, $\exists\, F \in \mathrm{Perf}(X)$ st. $D(X) \simeq \mathrm{Mod}_{R\mathrm{End}(F)}(\mathrm{dgMod}_k)$

Consequences:

1) (Easy) $\exists$ long exact sequence

$$0 \to \mathcal{O}(S)^\times \to \mathcal{O}(U)^\times \oplus \mathcal{O}(T)^\times \to \mathcal{O}(V)^\times \to$$
$$\to d\mathrm{Pic}(S) \to d\mathrm{Pic}(U) \oplus d\mathrm{Pic}(T) \to d\mathrm{Pic}(V) \to$$
$$\to d\mathrm{Br}(S) \to d\mathrm{Br}(U) \oplus d\mathrm{Br}(T) \to d\mathrm{Br}(V)$$

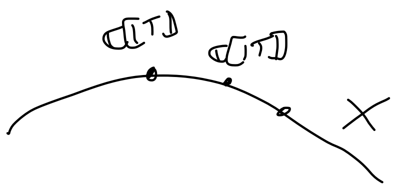
2) (Ongoing) Adelic descent

Basic idea: X is a curve

Thm (Weil)

$$\mathrm{Bun}_G(X) \simeq \left[G(\mathcal{O}_X) \backslash G(\mathbb{A}_X) / G(\mathbb{A}_X) \right]$$

$\uparrow$ completely decomposed in terms of formal completion at every point of X.



To give a vector bundle on X you only need to give a v.b. on each formal completion + some gluing data.

Conj. $dAz(X)$ satisfies adelic descent.

Beilinson higher adèles

Concrete gain:

$$dAz(X) \simeq \lim_n dAz(\mathbb{A}_X^n)$$

$\Downarrow$

new spectral sequence computing $d\mathrm{Br}(X) \rightsquigarrow$ new filtration $d\mathrm{Br}(X)$