# Nottingham 31/03/22_Full exceptional collections for anticanonical log Del Pezzo surfaces (joint with G.Gugiatti)

04 February 2022 21:37

$$X := X_{8k+4} \subseteq \mathbb{P}(2, 2k+1, 2k+1, 4k+1) \qquad (k \geq 1)$$

$$\begin{tikzcd} X \\ \mathcal{X} \arrow[u, squiggly] \end{tikzcd}$$

$\mathcal{X}$ canonical stack. (smooth) $\rightsquigarrow$ $\boxed{D^b(Coh(\mathcal{X}))}$

Why?

① All but finite examples of quasismooth, well-formed anticanonical log dP (JK'01)

↳ $-K_X \sim O(1)$

cyclic quotient sing's.

$\mathcal{X}$ smooth, orbifold

② Homological M.S.:

- $|-K| = \emptyset$ $\rightsquigarrow$ no way to run the intrinsic M.S. machinery (Gross-Siebert)
- $\check{X}$ known (Conti-Gugiatti): future goal

$$D^b(Coh(\mathcal{X})) \simeq FuK(\check{X})$$

**Thm** (Gugiatti-R.):
$D^b(Coh(\mathcal{X}))$ admits a full exceptional collection.

Recall

- $E \in D^b(Coh(\mathcal{X}))$ exceptional if $Hom^\bullet(E,E) = \mathbb{C} \cdot id$
- $(\underset{**}{E_1, \dots, E_m})$ except. coll if 1. $E_i$ exceptional
  2. $Hom^\bullet(E_i, E_j) = 0 \ \forall \ i > j$
- full: smallest $\Delta$'d cat containing the $E_i$ is $D^b(Coh(\mathcal{X}))$

**Cor:** $D^b(Coh(\mathcal{X})) \simeq D^b(mod\text{-}\Lambda)$

dg-algebra

(in fact make $\Lambda$ into a honest algebra)

---

**Main ingredient:** GL(2) McKay correspondence:

sing's of X are locally $\mathbb{C}^2/G$ (IU '11) ← Ishii, Ueda

finite, small, cyclic $\subseteq GL(2)$.

$\Longrightarrow$ $\exists$ a semiorthogonal decomposition:

$$D^b(Coh(X)) = \langle \underbrace{\overset{Ⓑ}{e_1, \dots, e_n}}_{\text{exc. coll. (from Sing X)}}, \overset{\text{fully faithful}}{\phi}(D^b(Coh(\overset{Ⓐ}{\tilde{X}}))) \rangle$$

min. resol. of $\tilde{X}$

Ⓒ: $Hom(e_i, \phi(D^b(Coh(X))))$

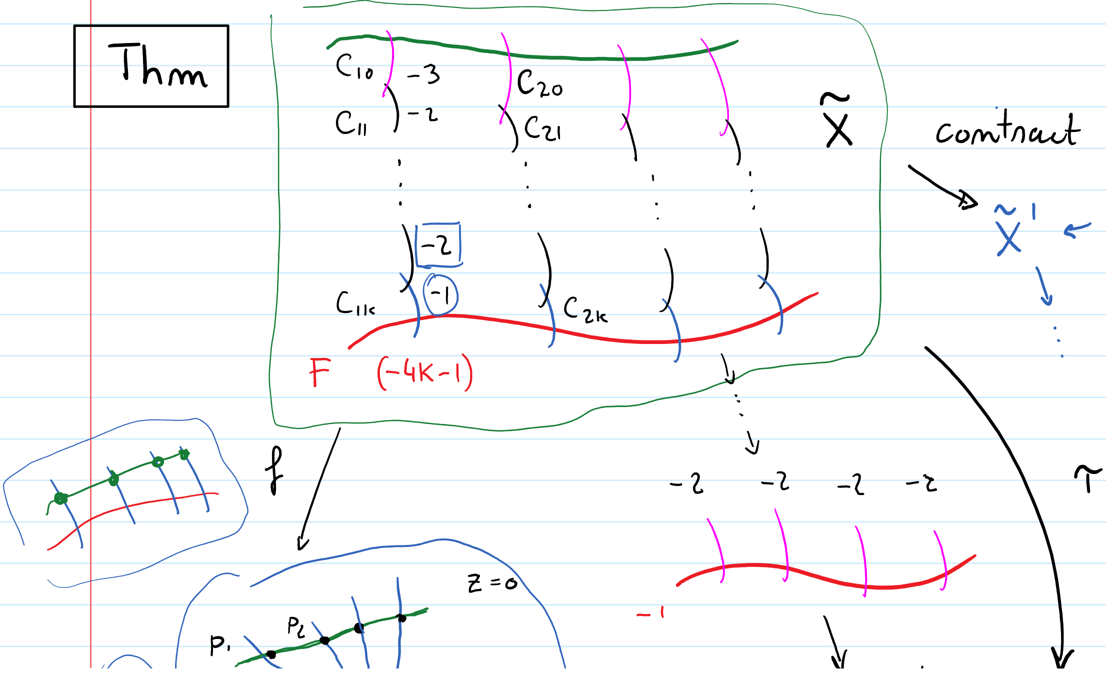
---

## Ⓐ Minimal resolution:

Singularities:

- $p$ ↺ $\frac{1}{4k+1}(1,1) = \langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \mid \zeta^{4k+1} = 1 \rangle$
- $p_1, p_2, p_3, p_4$ ↺ $\frac{1}{2k+1}(1,k) = \langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^k \end{pmatrix} \mid \xi^{2k+1} = 1 \rangle$

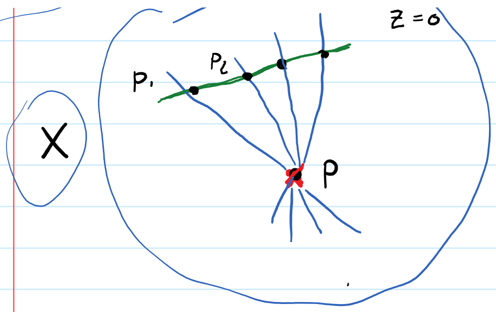
$\boxed{\text{Thm}}$

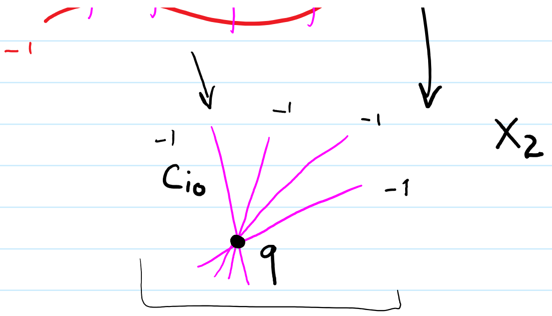


contract $C_{ik}$ ...

$\tilde{X}'$ ← min resol. for lower val's of k.

$$\frac{2k+1}{k} = [\underbrace{2, 2, .., 2}_{k-1}, 3]$$





where:
- $X_2$ is a del Pezzo surface of deg 2.
- $q \in X_2$ is a generalized Eckardt point.

Pf: toric geom. on $w\mathbb{P}^3$. $\rightsquigarrow X = \{ h^{(K)} = 0 \}$

$\Rightarrow X_2 = \{ h^{(0)} = 0 \}$

**Cor 1:** The moduli space of surfaces $X_{4K+8}$ is birat'l to the locus of $dP_2$ w/ an Eckardt pt (4 dim'l locus).

**Cor 2:** $D^b(\mathrm{Coh}(\tilde{X}))$ admits a full exc. collection.

Pf: $\tilde{X} \xrightarrow{\tau} X_2 \xrightarrow{\pi} \mathbb{P}^2$ sequence of blow-ups.

has exc. coll.

blowup formula (Orlov): $Bl_x(S) \xrightarrow{\pi} S$

$$D^b(Bl_x(S)) = \langle \pi^* D^b(S), \boxed{\mathcal{O}_E} \rangle$$

Caution: repeated blowups get complicated if the centers are on E.

---

Next: explicit version of Cor 2.

# Geometry of $dP_2$ surfaces:

1) $|-K| : X_2 \xrightarrow{2:1} \mathbb{P}^2 \supset R$ quartic curve.

56 line $\longleftarrow$ 28 bitangents.

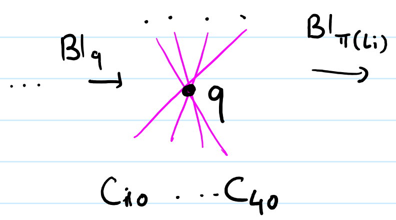
(-1)-curve smooth.

2) Any 7 disjoint (-1)-curves $L_i$ on $X_2$ define

$$\begin{tikzcd} X_2 \arrow[r, "\pi"] \arrow[d, phantom, "\parallel"] & \mathbb{P}^2 \\ Bl_{\pi(L_i)} \mathbb{P}^2 & \end{tikzcd}$$

To get a simpler collection:

$$\tilde{X} \longrightarrow X_2 \xrightarrow{\pi} \mathbb{P}^2$$



want: $\{L_1, \dots, L_7\} \cap \{C_{10}, \dots, C_{14}\} = \emptyset$

this can always be arranged.

lattice theory (MAGMA)

## Explicit Cor 2:

$$D^b(\tilde{X}) = \langle \overbrace{(\pi_1)^* D(\mathbb{P}^2)}, \overbrace{\mathcal{O}_{L_1}, \dots, \mathcal{O}_{L_7}}^{X_2 \to \mathbb{P}^2}, \tau^* \mathcal{O}_q, \overbrace{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4} \rangle$$

where $\mathcal{B}_i = (\mathcal{O}_{\sum_{j=1}^{k} C_{ij}}, \dots, \mathcal{O}_{C_{ik-1}+C_{ik}}, \mathcal{O}_{C_{ik}})$

Pf:

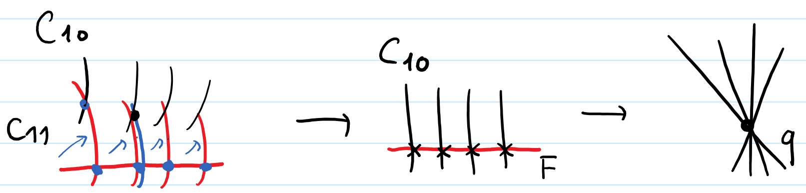
dP

$X_2 \xrightarrow{\pi} \mathbb{P}^2$

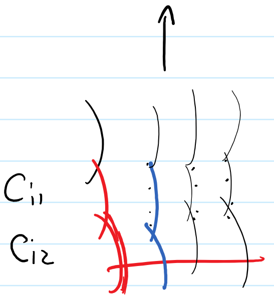
Pf:

dP

$X_2 \xrightarrow{\pi} \mathbb{P}^2$



$\left(\mathcal{O}_{F+\Sigma C_{i1}}, \underbrace{\mathcal{O}_{C_{11}},\dots,\mathcal{O}_{C_{14}}}\right)$ $\quad (\mathcal{O}_F)$ $\quad (\mathcal{O}_{L_i})$



$B_i$

$\left(\mathcal{O}_{F+2\Sigma C_{i2}+\Sigma C_{i1}}, \mathcal{O}_{C_{11}+C_{12}},\dots,\mathcal{O}_{C_{41}+C_{42}}, \mathcal{O}_{C_{12}},\dots,\mathcal{O}_{C_{42}}\right)$ ▨

---

Now we move to the stack: Ⓐ

$$\to D^b(\mathcal{X}) = \langle \underbrace{\mathcal{E}_p, \mathcal{E}_{p_1},\dots,\mathcal{E}_{p_4}}, \underbrace{\Phi D^b(\tilde{X})} \rangle$$

## Ⓑ Singularity contributions:

[IU'11] · $\mathcal{E}_p = (\mathcal{O}_p\otimes\underbrace{\sigma_2},\dots,\mathcal{O}_p\otimes\underbrace{\sigma_{4k}})$ ↙ $\frac{1}{4k+1}(1,1)$ : all nonspecial rep's of G

· $\mathcal{E}_{p_i} = (\mathcal{O}_{p_i}\otimes\underbrace{\rho_{k+1}},\dots,\underbrace{\mathcal{O}_{p_i}\otimes\underbrace{\rho_{2k}}})$

$\mathrm{Hom}^\bullet(\mathcal{O}_p\otimes\rho, \mathcal{O}_p\otimes\sigma)$ ↑ ↑ (Shur's Lemma)

⇝ maps within $\mathcal{E}_p, \mathcal{E}_{p_i}$ given by McKay quiver. (✓)

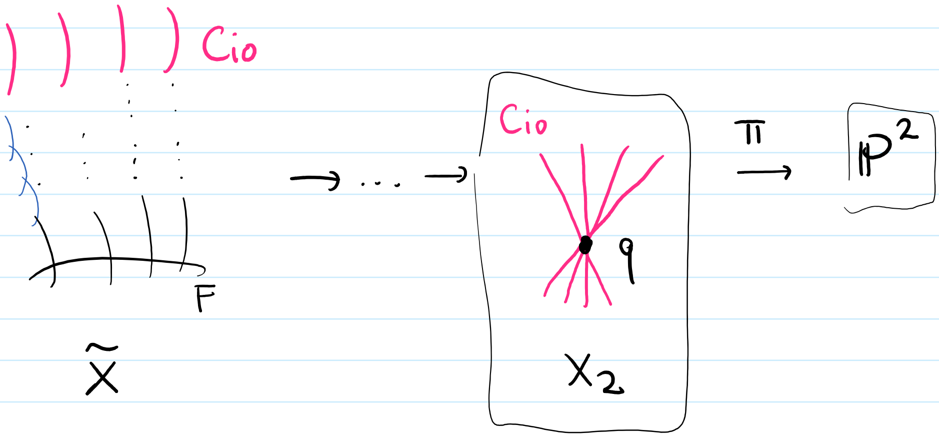
## Ⓒ maps $\mathcal{E}_p. \mathcal{E}_{p_i} \to \Phi D(\tilde{X})$

Apply the left adjoint $\Psi$ & compute on $\tilde{X}$.

(we compute $\Psi(\mathcal{O}_p\otimes\rho)$ using ↗ AR sequences ↘ toric geometry) ⇝ Ishii

⇝ interesting wrinkle:

$$\psi(\mathcal{O}_{p_i} \otimes \rho_{2k}) = \mathcal{O}_{C_{i0}}(-2)$$



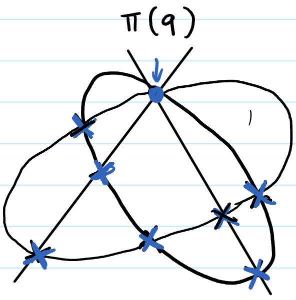
Q: Can we control $\pi(C_{i0})$?

**Prop (Classification of gen'd Eckardt pts):**

$q \in X_2$ Eckardt, $\pi : X_2 \to \mathbb{P}^2$, blowup at $x_1, \ldots, x_7$.

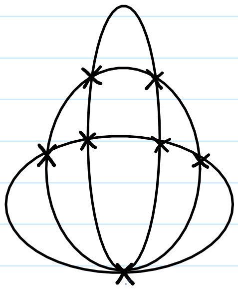
Then the only possibilities are:

①



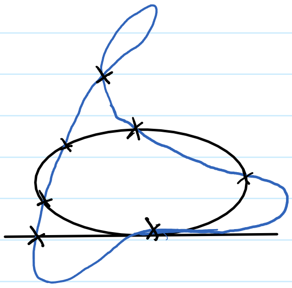
$\pi(C_{i0})$ line $(i=1,2)$
conic $(i=3,4)$

②



3 conic
1 contracted

③



1 line 1 cubic
1 conic 1 contr.

↑ can always arrange this. (lattice thy)

---

**Next:**
- mirror symmetry: compute Fukaya of mirrors & compare w/ the algebra $\Lambda$
- other applications:
  - HMS for other surfaces
  - stability conditions & moduli