Boundary divisors in the compactification by stable surfaces of moduli of Horikawa surfaces

University of Nottingham – Algebraic Geometry Seminar

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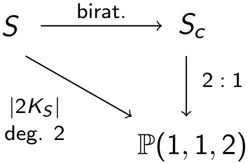
Horikawa surface

Definition

A **Horikawa surface** S is a smooth projective connected minimal surface over $\mathbb{C}$ which is of general type and satisfying

$$K_S^2 = 1, \qquad p_g = h^2(\mathcal{O}_S) = 2, \qquad q = h^1(\mathcal{O}_S) = 0.$$



- S_c = canonical model of S. (K_{S_c} ample, ADE sing's.)
- $S_c \to \mathbb{P}(1,1,2)$ is a $2:1$ cover branched along $V(F_{10}(x,y,z))$.
- Hence, $S_c = V(w^2 - F_{10}(x,y,z)) \subseteq \mathbb{P}(1,1,2,5)$.

Moduli of Horikawa surface

Theorem (Gieseker, 1977)

There exists a quasi-projective coarse moduli space $\mathbf{M}_H$ parametrizing the canonical models S_c of Horikawa surfaces.

Remark

The dimension of $\mathbf{M}_H$ can be computed as follows:

$$\begin{aligned} \dim(\mathbf{M}_H) = 36 & \quad (\#\text{monomials } x^a y^b z^c \text{ s.t. } a+b+2c=10) \\ -1 & \quad (\text{projective scaling}) \\ -7 & \quad (\dim \mathrm{Aut}(\mathbb{P}(1,1,2)) \\ = 28. & \end{aligned}$$

Problem today. Compactify $\mathbf{M}_H$.

The problem of compactifying $\mathbf{M}_H$

- **Hodge Theory.** The period domain for Horikawa surfaces is **not** Hermitian symmetric $\implies$ Baily–Borel and toroidal compactification methods do not apply!

  Remark. Work of Kato–Usui and Green–Griffiths–Laza–Robles towards generalizing these techniques in the non-Hermitian symmetric case.

- Other compactifications of $\mathbf{M}_H$ attracted attention:
  - **Geometric Invariant Theory.** $\mathbf{M}_H \subseteq \overline{\mathbf{M}}_H^{\mathrm{GIT}}$ (Wen, 2021);
  - **Minimal Model Program.** $\mathbf{M}_H \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$ due to Kollár, Shepherd-Barron, Alexeev.

    Roughly, it is the analogue of the Deligne–Mumford, Knudsen comp's $\overline{\mathrm{M}}_{g,n}$ for moduli of higher dim. alg. var's.

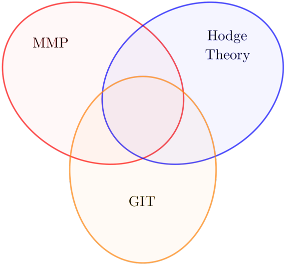
KSBA compactification

For a moduli space $\mathbf{M}$, the compactification $\overline{\mathbf{M}}^{\mathrm{KSBA}}$ is

- geometric (paramet. **stable varieties** or **stable pairs**);
- modular (coarse moduli space);
- expected to have rich boundary structure.

Problems to investigate.

- In general, $\partial\overline{\mathbf{M}}^{\mathrm{KSBA}}$ unknown, hard to study (MMP);
- Relation with other compactification methods?



KSBA compactification

Definition

X variety, D $\mathbb{Q}$-divisor with coefficients in $[0,1]$. (X, D) is **stable** if

- (X, D) is semi-log canonical;
- $K_X + D$ is ample.

If $(X, 0)$ is stable, then we say that X is a **stable variety**.

Theorem (Kollár, Shepherd-Barron, Alexeev, ...)

There exists a projective coarse moduli space parametrizing stable pairs (X, D) with certain fixed numerical invariants.

Example

S_c canonical model of a smooth Horikawa surface S.

- S_c is a stable surface;
- $\overline{\mathbf{M}}_H^{\mathrm{KSBA}} := \{S_c \text{ and their stable degenerations}\}$.

$\rightarrow$ Irred. component parametrizing smooth Horikawa surfaces.

$\overline{\mathbf{M}}_H^{\mathrm{KSBA}}$: What is known

- Franciosi–Pardini–Rollenske (2017):

$$\mathbf{M}_H \subsetneq \mathbf{M}_H^{\mathrm{Gor}} \subsetneq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$$

$\mathbf{M}_H^{\mathrm{Gor}} :=$ parametrizes stable surfaces with Gorenstein singularities. They prove that:

  - $\dim \partial \mathbf{M}_H^{\mathrm{Gor}} = 20$, not pure.
  - $\mathbf{M}_H^{\mathrm{Gor}}$ parametrizes irreducible stable surfaces with at worst elliptic singularities.

- Franciosi–Pardini–Rana–Rollenske (2022):
  - $\mathbf{D}_1, \mathbf{D}_2 \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$ divisors parametrizing irreducible stable surfaces with a unique $\frac{1}{4}(1,1), \frac{1}{18}(1,5)$ singularity.

Our work is complementary to the above.

$\overline{\mathbf{M}}_H^{\mathrm{KSBA}}$: Main results

Let $\Sigma \in \{E_{12},\ E_{13},\ E_{14},\ Z_{11},\ Z_{12},\ Z_{13},\ W_{12},\ W_{13}\}$.
These indicate certain non-log canonical isolated surface singularities (they will be described later).

$\overline{\mathbf{M}}_H^{\mathrm{KSBA}}$: Main results

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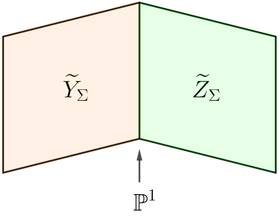
Theorem (Gallardo–Pearlstein–S–Zhang, 2022)

(i) *$\exists$ 8 boundary divisors $\mathbf{D}_\Sigma \subseteq \overline{\mathbf{M}}^{\mathrm{KSBA}}$, $\mathbf{D}_\Sigma \neq \mathbf{D}_1, \mathbf{D}_2$. $\mathbf{D}_\Sigma$ generically parametrizes stable surfaces $S_\Sigma = \widetilde{Y}_\Sigma \cup \widetilde{Z}_\Sigma$.*

(ii) *$\overline{\mathbf{M}}_H^{\mathrm{KSBA}} \dashrightarrow \overline{\mathbf{M}}_H^{\mathrm{GIT}}$, given by the identity of $\mathbf{M}_H$, extends to the interior of $\mathbf{D}_\Sigma$ mapping to orbits of stable points.*

(iii) *The limiting mixed Hodge structure of $S_\Sigma = \widetilde{Y}_\Sigma \cup \widetilde{Z}_\Sigma$ is pure.*

$\widetilde{Y}_\Sigma =$ ADE K3 surface in weighted projective space (8 of the 95 families in Miles Reid's list)



$\widetilde{Z}_\Sigma$ has finite quotient sing's $h^2(\mathcal{O}) = 1,\ h^1(\mathcal{O}) = 0$ birational to a K3 surface in 5 of the 8 cases

Construction of the divisors $\mathbf{D}_\Sigma \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$

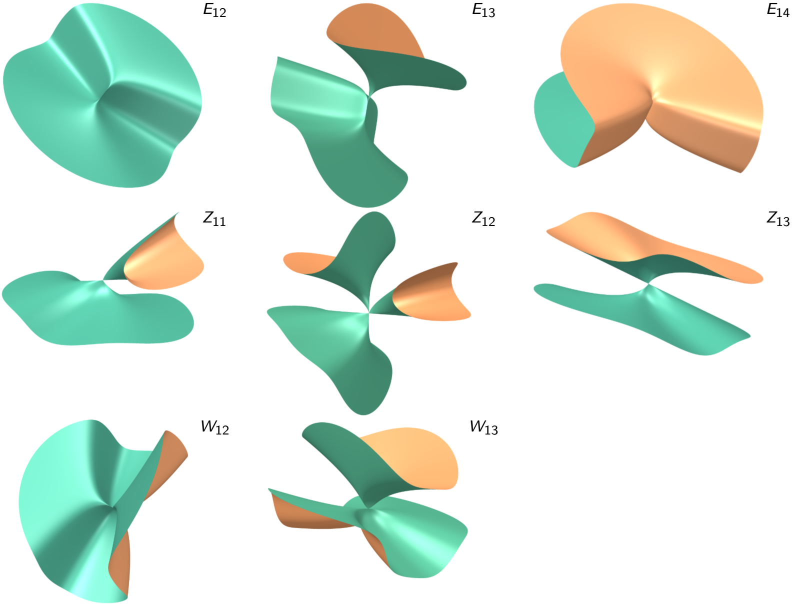
Question. What are the simplest **not log canonical** isolated singularities that a Horikawa surface can acquire degenerating?

- Consider isolated surface singularities of modality 1 which are **not log canonical** and that can be realized at

$$[1:0:0:0] \in S_0 = V(w^2 - F_{10}(x,y,z)) \subseteq \mathbb{P}(1,1,2,5).$$

- There are eight such singularities with germs in $\mathbb{A}^3_{y,z,w}$ given by

E_{12}	$w^2 = z^3 + y^7$	E_{13}	$w^2 = z^3 + y^5 z$
E_{14}	$w^2 = z^3 + y^8$	Z_{11}	$w^2 = yz^3 + y^5$
Z_{12}	$w^2 = yz^3 + y^4 z$	Z_{13}	$w^2 = yz^3 + y^6$
W_{12}	$w^2 = z^4 + y^5$	W_{13}	$w^2 = z^4 + y^4 z$



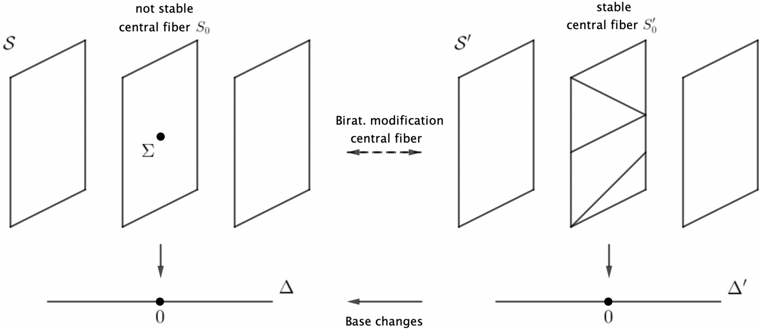
Pictures realized using the software SURFER from `www.imaginary.org`

Construction of the divisors $\mathbf{D}_\Sigma \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$: general strategy

- For each Σ, we:
  - Find the general 1-parameter smoothing

$$S_0 \subseteq \mathcal{S} \to \Delta = \mathrm{Spec}(\mathbb{C}[[t]]).$$

  - Compute the **stable replacement** $S_0' \subseteq \mathcal{S}' \to \Delta'$ of the central fiber $S_0 \subseteq \mathcal{S} \to \Delta$.



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Definition

$\mathbf{D}_\Sigma \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$ is the Zariski closure of the subset of points parametrizing stable surfaces S_0' as above.

  - Show that $\mathbf{D}_\Sigma \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$ is 27-dimensional, hence a divisor.

Remark. If $S_0 \subseteq \mathcal{S} \to \Delta$ is just any smoothing, then in general it could be quite hard to find the stable replacement $S_0' \subseteq \mathcal{S}' \to \Delta'$.

Strategy. We construct specific 1-parameter smoothings $S_0 \subseteq \mathcal{S} \to \Delta$ for which we show that:

- the stable replacement $S_0' \subseteq \mathcal{S}' \to \Delta'$ is obtained after a single weighted blow up $\mathcal{S}' \to \mathcal{S}$ and no base changes, so $\Delta' = \Delta$.
- S_0' depend on 27 parameters. So $\mathbf{D}_\Sigma \subseteq \overline{\mathbf{M}}_H^{\mathrm{KSBA}}$ is a divisor.

In the next slide we illustrate the construction of $S_0 \subseteq \mathcal{S} \to \Delta$.

The 1-parameter smoothings $S_0 \subseteq \mathcal{S} \to \Delta$

(1) Introduce $\mathrm{wt}_\Sigma \colon \{x^a y^b z^c \mid a+b+2c=10\} \to \mathbb{Z}$ such that: for general $h(x,y,z)$ of degree 10, if $h = h_- + h_0 + h_+$, then

$$S_0 := V(w^2 - (h_0 + h_+)) \subseteq \mathbb{P}(1,1,2,5)$$

has precisely one singularity at $[1:0:0:0]$, and this is of type Σ.

Example. $\Sigma = E_{12}$. Recall the germ is $w^2 = z^3 + y^7$. Define

$$\mathrm{wt}_\Sigma(x^a y^b z^c) = 6b + 14c - 42.$$

Note that $x^4 z^3$ and $x^3 y^7$ are the only monomials of weight zero.

(2) If $\mathrm{wt}_\Sigma(x^a y^b z^c) < 0$, then let

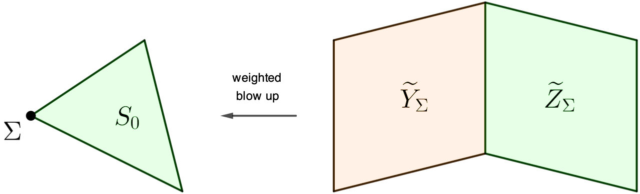
$$t \star x^a y^b z^c := t^{-\mathrm{wt}_\Sigma(x^a y^b z^c)} x^a y^b z^c.$$

We define $t \star h_-$ extending by linearity.

(3) $\mathcal{S} := V(w^2 - (t \star h_- + h_0 + h_+)) \subseteq \mathbb{P}(1,1,2,5) \times \Delta$.

The stable replacements $S_0' \subseteq \mathcal{S}' \to \Delta'$

Consider $\mathcal{S}' \to \mathcal{S}$ appropriate weighted blow up at $[1:0:0:0]$:



- $\widetilde{Y}_\Sigma :=$ exceptional divisor of $\mathcal{S}' \to \mathcal{S}$.
- $\widetilde{Z}_\Sigma :=$ strict transform of S_0.

Example. $\Sigma = E_{12}$.

$$\begin{array}{ccccc} \mathcal{S} & \longleftarrow & \mathcal{S}' & \supseteq & \widetilde{Y}_{E_{12}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{P}(1,1,2,5) \times \Delta & \xleftarrow[(t^{42},\, y^7,\, z^3,\, w^2)]{} & \mathcal{X}' & \supseteq & \mathbb{P}(1,6,14,21) \end{array}$$

$\widetilde{Y}_{E_{12}} \subseteq \mathbb{P}(1,6,14,21)$ degree 42 hypersurface, hence it is an ADE K3.

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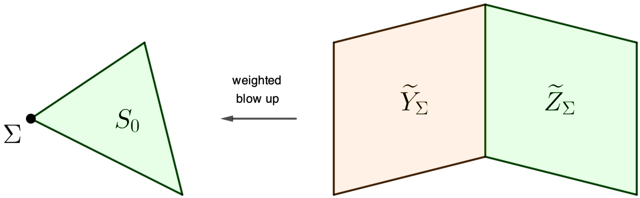

- $\widetilde{Y}_\Sigma :=$ exceptional divisor of $\mathcal{S}' \to \mathcal{S}$.
- $\widetilde{Z}_\Sigma :=$ strict transform of S_0.

The very first result that we prove is then

Theorem (Gallardo–Pearlstein–S–Zhang, 2022)

The reducible surface $\widetilde{Y}_\Sigma \cup \widetilde{Z}_\Sigma$ is stable.

Thank you for your attention!