BIRATIONAL MAPS OF SEVERI-BRAUER SURFACES, WITH APPLICATIONS TO CREMONA GROUPS OF HIGHER RANK

joint with JÉRÉMY BLANC and EGOR YASINSKY

$\mathrm{Bir}_k(\mathbb{P}^n) = \{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \text{ bir defd over } k\}$

Cremona group of rank n over the field k

$\mathrm{Bir}(X)$ X smooth proj var

Q1 Is $\mathrm{Bir}(X)$ generated by elements of finite order?

Q2 Large quotients of $\mathrm{Bir}(X)$?

i.e. gp hom

$\mathrm{Bir}(X) \twoheadrightarrow G$

"large"

- G abelian ⇝ abelianization of $\mathrm{Bir}(X)$?
- kernel = normal subgp

For $X = \mathbb{P}^n / k$:

$n=2$ $\quad k=\bar{k}$ $\quad$ Thm (Noether-Castelnuovo)

$$\mathrm{Bir}_k \mathbb{P}^2 = \langle PGL_3(k), \sigma \rangle$$

YES

$$\sigma : (x,y) \mapsto \left(\tfrac{1}{x}, \tfrac{1}{y}\right)$$ involution

gen. by involutions

k perfect $\quad$ gen. by involutions [Lamy-S.]

Construction of quotients:

$$G = * \oplus \mathbb{Z}/2\mathbb{Z}$$

k perfect

$k \neq \bar{k}$

[Zimmermann, Lamy-Zimmermann, S.]

$n=3$ $\quad$ NO $\quad$ [Shinder-Lin]

$k=\mathbb{Q}$ $\quad G = \mathbb{Z}$

$n \geq 4$ also for $k=\mathbb{C}$ $\quad$ NO

other constructions:

$$G = * \oplus \mathbb{Z}/2$$

[Blanc-Lamy-Zimmermann, Zikas, Blanc-Yasinski]

$\Downarrow$

$\exists$ non-trivial normal subgps of the Cremona group of rank $n \geq 3$

What if X is geometrically rational?

Thm A (BSY) Let S be a non-trivial Severi-Brauer surface (i.e. $S_{\bar{k}} \simeq \mathbb{P}^2_{\bar{k}}$ but $S_k \not\simeq \mathbb{P}^2_k$).

Set $\mathcal{P}_d := \{\text{points on } S \text{ of degree } d\}/\mathrm{Aut}(S)$.

Then there exists a surj gp homo

$$\mathrm{Bir}(S) \longrightarrow \bigoplus_{\mathcal{P}_3\setminus\{p_0\}} \mathbb{Z}/3 * \left(\mathop{*}_{\mathcal{P}_6} \mathbb{Z}\right)$$

for any $p_0 \in \mathcal{P}_3$.

In particular, if $\mathcal{P}_6 \neq \emptyset$ then $\mathrm{Bir}(S)$ is not generated by elts of finite order.

- For two pts p, q on S of degree 3: [Lemma]

  $p \underset{\mathrm{Aut}(S)}{\sim} q \iff p$ and q have same splitting field

  $\iff p, q$ have k-isom. residue fields

- $|\mathcal{P}_3| \geq 2$ so gp homo is not trivial

- Shramov studied finite subgps of $\mathrm{Bir}(S)$.

  $\rightsquigarrow$ no involutions !

Thm B (BSY) $\forall n \geq 4$:

$\exists \ \mathrm{Bir}_{\mathbb{C}}\mathbb{P}^n \twoheadrightarrow *_{|\mathbb{C}|} \mathbb{Z}$

large!

In particular, for any group G with $|G| \leq |\mathbb{C}|$, $\exists$ $\mathrm{Bir}_{\mathbb{C}}\mathbb{P}^n \twoheadrightarrow G$.

From Thm A to Thm B:

idea

X s.t. general fiber is a
$\downarrow$ non-trivial Severi-Brauer
B surface over $\mathbb{C}(B)$.

Fact if B curve, then $\mathbb{C}(B)$ is C_1 and so $\nexists$ non-trivial Severi-Brauer surface over $\mathbb{C}(B)$.

$\rightsquigarrow$ need $\dim B \geq 2$

$\Rightarrow \dim X = \dim B + 2 \geq 4$.

need: X rational

Also studied by Maeda, Kresch-Tschinkel

$\downarrow$

examples of such vars that are not even stably rational

Strategy:

need generators & relations

groupoid:

$$\mathrm{BirMori}(X) = \{ f : X_1 \overset{\text{bir}}{\dashrightarrow} X_2 \mid X_1, X_2 \text{ are Mfs bir. to } X \}$$

Thm (Corti, Iskovskikh, Hacon-McKernan)
$\mathrm{BirMori}(X)$ is generated by Sarkisov links

for surfaces:

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_1 \arrow[rr, dashed] & & X_2 \end{tikzcd}$$

blow-up of at most one closed pt.

From now: surfaces

k perfect field
↳ e.g. $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{C}(t_1, t_2)$

Def. X surface, $\pi : X \to B$ is a rank r fibration if π surj with connected fibers and

- $\dim B < \dim X = 2$
- X, B smooth
- $\rho(X/B) = r$
- $-K_X$ π-ample

[$-K_X \cdot C > 0$ $\forall$ curves C that are contracted by π]

if B is a point: X del Pezzo surface!

__Rmk__ · if $r=1 \longleftrightarrow$ Mfs (Mori fiber space)

· $r=2 \longleftrightarrow$ Sarkisov links

· $r=3 \longleftrightarrow$ relations between Sarkisov links.

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] \arrow[ddd] & \\
X_1 \arrow[rr, dashed] \arrow[d, "\mathrm{Mfs}"'] & & X_2 \arrow[d, "\mathrm{Mfs}"] \\
B_1 \arrow[dr] & & B_2 \arrow[dl] \\
 & B &
\end{tikzcd}$$

rank 2 fibration.

__Facts__ S non-trivial SB surf.

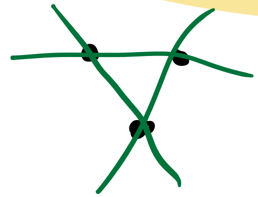
- p closed pt on $S \Rightarrow 3 \mid \deg(p)$
- 1–1 corresp. btw SB var and central simple algebras
  $\rightsquigarrow S^{op}$ opposite Severi–Brauer surface, $S^{op} \not\cong S$

$$\begin{tikzcd}
 & Y \arrow[dl, "3"'] \arrow[dr] & \\
S \arrow[rr, dashed] & & S^{op}
\end{tikzcd}$$

contract 3 lines

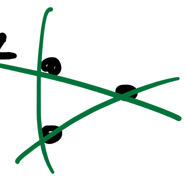
$$\begin{tikzcd}
 & Y \arrow[dl, "6"'] \arrow[dr] & \\
S \arrow[rr, dashed] & & S^{op}
\end{tikzcd}$$

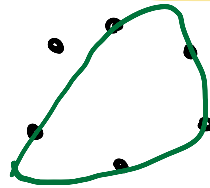
contract 6 conics



12 E

$\mathbb{P}^2$

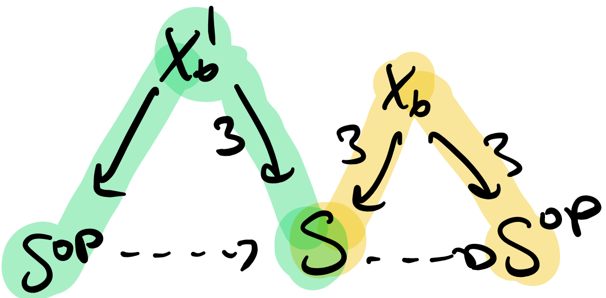


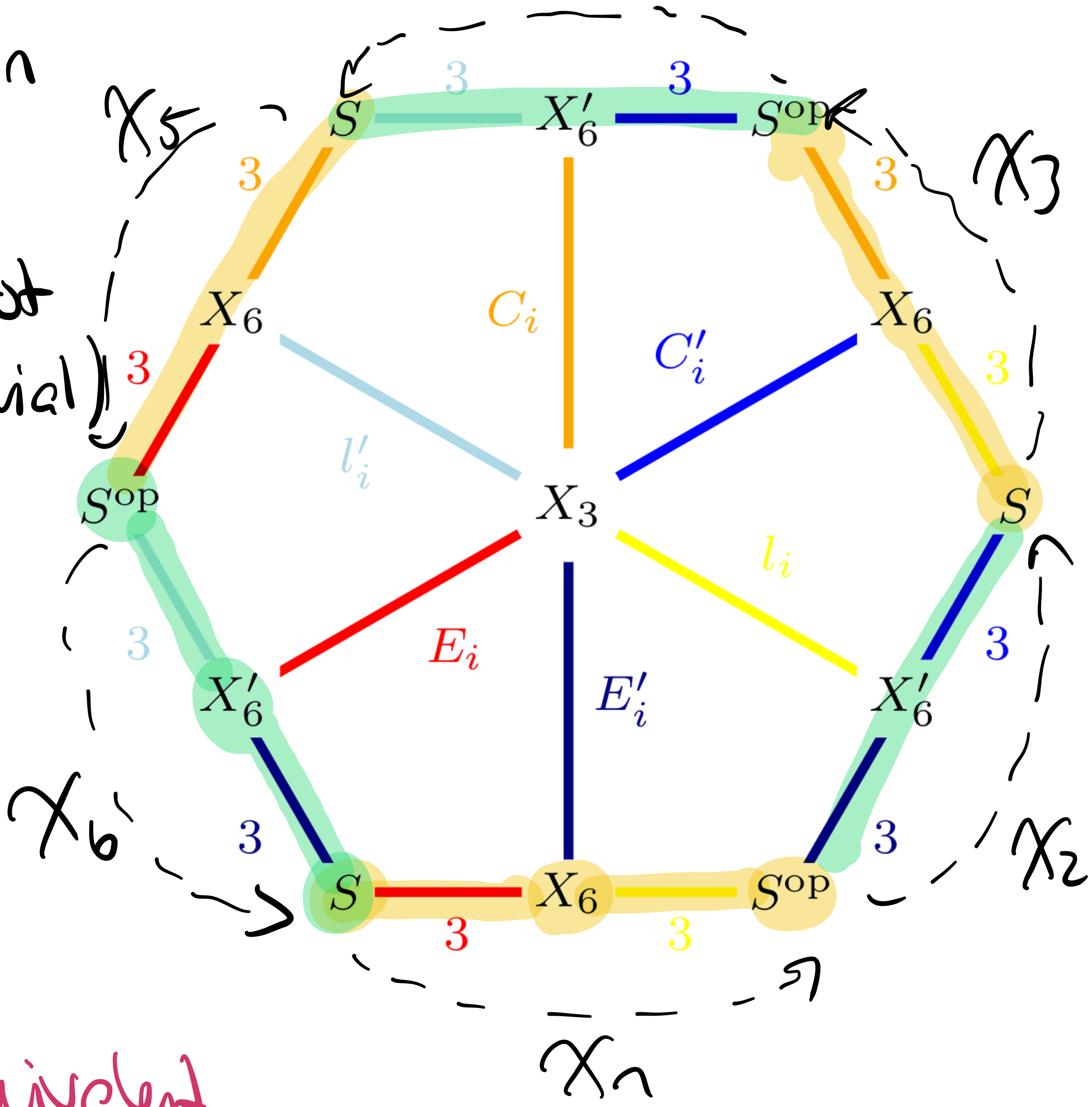


C_i conic

χ_2

Because rank r fibration over a pt have to be delPezzo, links $S \dashrightarrow S^{op}$ do not appear in any (non-trivial) relation!





X_1, X_3, X_5 are equivalent

X_2, X_4, X_6 are equivalent

Thm S non-trivial SB surface over a perfect field

$$E_d := \{ \chi : S \overset{d}{\dashrightarrow} S^{op} \text{ Sarkisov link} \} / \sim$$

$\chi \sim \chi'$ if $\exists \alpha, \beta$ iso

$$\begin{tikzcd} S \arrow[r, dashed, "\chi"] \arrow[d, "\alpha"'] & S^{op} \arrow[d, "\beta"] \\ S \arrow[r, dashed, "\chi'"] & S^{op} \end{tikzcd}$$

$$\exists \ \mathrm{Bir}\,\mathrm{Mori}(S) \twoheadrightarrow \bigoplus_{E_3} \mathbb{Z}/3 * \left(\mathop{*}_{E_6} \mathbb{Z} \right)$$

$$\chi \text{ Sarkisov link} \longmapsto \begin{cases} 1_{[\chi]} & \chi : S \dashrightarrow S^{op} \\ -1_{[\chi^{-1}]} & \chi : S^{op} \dashrightarrow S \end{cases}$$

χ is not equivalent to χ^{-1}