Logarithmic Toric Quasimaps

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Let's think about curves in projective space ($C \hookrightarrow \mathbb{P}^N$)
Here are 3 perspectives:

1) via the ideal
2) via the parametrisation
3) via the defining line bundle & sections
   $(\mathcal{L}, u_0, \ldots, u_N)$

For smooth embedded curves not really any difference.
2) & 3) are literally the same.

But in the world of modern curve counting they give different approaches...

The modern curve counting machine:

smooth proj. variety		"Space of curves in X"		invariants
X	$\rightsquigarrow$	$\mathcal{M}(X)$	$\longrightarrow$	$\int_{\mathcal{M}(X)} \cdots$
				"curve counts"

$\mathcal{M}(X)$ needs to be
i) proper
ii) (virtually) smooth

1) $\rightsquigarrow$ Donaldson-Thomas Theory
2) $\rightsquigarrow$ Gromov-Witten Theory $\overline{\mathcal{M}}_{g,n}(X,\beta)$
3) $\rightsquigarrow$ Quasimap Theory $Q_{g,n}(X,\beta)$

$g \rightsquigarrow$ genus
$\beta \rightsquigarrow$ degree
$n \rightsquigarrow$ # markings

How do 2) & 3) differ?

$\overline{M}_{g,n}(\mathbb{P}^N,d) = \{(C,p_1,\dots,p_n,f: C\to\mathbb{P}^N)\}$

$Q_{g,n}(\mathbb{P}^N,d) = \{(C,p_1,\dots,p_n,\mathcal{L},u_0,\dots,u_N)\}$

But the limiting objects differ. In particular, in $Q_{g,n}(\mathbb{P}^N,d)$ $u_0,\dots,u_N$ can simultaneously vanish at finitely many points in C.

Example: $\overline{M}_{0,*}(\mathbb{P}^2,2)$ vs $Q_{0,*}(\mathbb{P}^2,2)$: $\forall t\in\mathbb{C}^*$,

$(\mathbb{P}^1\to\mathbb{P}^2, [z_0:z_1]\mapsto[tz_0^2:z_0z_1:z_1^2])\in\overline{M}_{0,*}(\mathbb{P}^2,2)$

"

$(\mathbb{P}^1, \mathcal{O}(2), tz_0^2, z_0z_1, z_1^2)\in Q_{0,*}(\mathbb{P}^2,2)$

what happens as $t\to 0$?

setting $t=0$ gives $[z_0:z_1]\dashrightarrow[0:z_0z_1:z_1^2]$

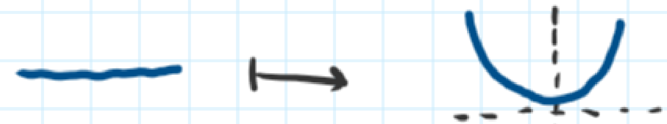
not defined at $[1:0]$...

on the other hand $(\mathbb{P}^1,\mathcal{O}(2),0,z_0z_1,z_1^2)\in Q_{0,*}(\mathbb{P}^2,2)$

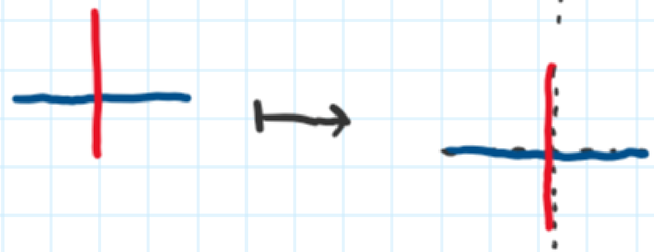
(only finitely many, 1, basepoint)

What about the limit in $\overline{M}_{0,*}(\mathbb{P}^2,2)$?

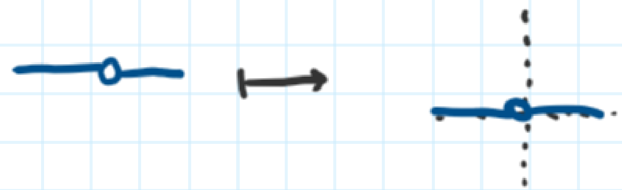
(Q & $\overline{M}$) $t\neq 0$



($\overline{M}$) $t=0$



(Q) $t=0$



Quasimap space is a more efficient compactification...

Quasimap Summary [CFKM]

- X a nice GIT quotient $\Rightarrow \exists\ Q_{g,n}(X,\beta)$ proper DM stack
- $Q_{g,n}(X,\beta)$ admits a virtual class
- Quasimap invariants easier to compute
- $\underbrace{\text{wall crossing formulas}}_{\text{mirror map}}$ GW inv to Quasimap invariants

Relative Curve Counting

Fix not just X but also $D \subseteq X$ divisor.

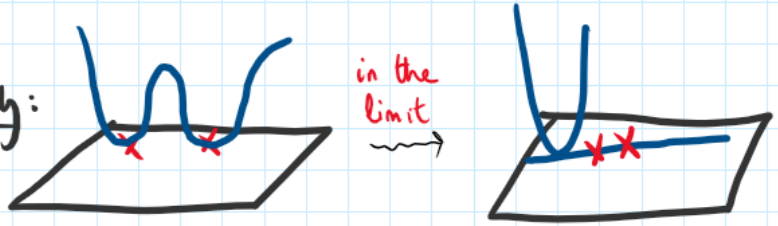
Now think about curves in X w/ fixed tangency to D

(e.g. conics in $\mathbb{P}^2$ tangent to a line)

How do you build an appropriate moduli space?

what's the problem?

generically:



so the space of curves in X tangent to D w/ the appropriate contact orders at the markings is not compact!

in the Gromov-Witten setting there have been many iterations of sol'n.

Most general due to: Abramovich-Chen-Gross-Siebert

using logarithmic structures

extra data on curve and target so you can measure "tangency" even if the curve falls into D.

Logarithmic Quasimaps

Simpler solution [Battistella-Nabijou] following [Gathmann].

Take the closure of locus w/ correct tangencies in ordinary quasimap moduli space. (restrictive, $g=0$, D smooth & v. ample)

Instead want to utilise the modern solution in GW Theory.

What is tangency?

$f : (C, p) \to (X, D)$ (s_D (local eqn) for D)

tangency at p = order of vanishing of $f^* s_D$ at p.

E.g. $f : (\mathbb{P}^1, [0:1]) \to (\mathbb{P}^2, \{x_0 = 0\})$ ($f^* x_0 = z_0^2$)

$[z_0 : z_1] \longmapsto [z_0^2 : z_0 z_1 : z_1^2]$ has tangency 2 at $[0:1]$

For quasimaps there is no map but we still have $(\mathcal{L}_D, u_D)$ on the curve so tangency is $\mathrm{ord}_p u_D$

(same problem in the limit; $u_D \equiv 0$ & so $\mathrm{ord}_p u_D = \infty$)

want to endow C & X w/ logarithmic structures (extra data) to keep track of this...

but there is no map $f : C \dashrightarrow X$

Fact/Analogy

$C \longrightarrow \mathbb{A}^1 \quad \longleftrightarrow \quad$ regular function

$C \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] \quad \longleftrightarrow \quad$ line bundle/section pair $(\mathcal{L}, u)$

now we have an actual map to put this extra data on...

<u>Def</u>$^{\underline{n}}$: a logarithmic quasimap to $\mathbb{P}^N|H$ is $(C, p_1, \ldots, p_n, \mathcal{L}, u_0, \ldots, u_N)$ + a logarithmic enhancement of the map $C \xrightarrow[(\mathcal{L}, u_0)]{} [\mathbb{A}^1/\mathbb{G}_m]$ s.t. this map has the correct tangency orders.

$\Downarrow$

'<u>Theorem</u>': Let X smooth proj. toric variety, D any s.n.c divisor ($D = D_1 + \ldots + D_r$), g, n integers, β curve class on X & $(\alpha_i)_j$ contact order data s.t. $\sum_j (\alpha_i)_j = D_i \cdot \beta$. Then $\exists$ moduli space $Q^{\log}_{g,\alpha}(X|D,\beta)$ parametrising log quasimaps which is proper DM stack.

'<u>Theorem</u>': $\exists$ perfect obs theory on $Q^{\log}$ leading to a virtual fundamental class.

The invariants agree w/ [B-N] in $g=0$, D smooth v. ample.

The logarithmic/relative quasimap spaces are also simpler than their GW counterparts.

d	# boundary divisors	
	$Q^{\log}_{0,2}(\mathbb{P}^N\|H, d)$	$\overline{M}^{\log}_{0,2}(\mathbb{P}^N\|H, d)$
1	2	3
2	3	7
3	4	14
4	5	26
5	6	45
6	7	75

What is this good for?

① Relative/logarithmic wall crossing
② Local/logarithmic correspondence

① [FTY] show a generating function for relative GW invariants ($g=0$) can be obtained via a change of variables (mirror map) from a "relative I-function"

[Battistella-Nabijou] show this relative I-function is a generating function for relative quasimap invariants

Evidence for wall-crossing in the relative/logarithmic setting...

② [vGGR] show that maximal contacts log/relative invariants coincide w/ local invariants

(*) $\left[\overline{M}^{\log}_{0,(d)}(X|D,\beta)\right]^{vir} = (-1)^{d+1}\cdot d\cdot\left[\overline{M}_{0,0}(\mathcal{O}_X(-D),\beta)\right]^{vir}$

and conjectured a generalisation when $D = D_1+\dots+D_r$.

Although this holds in cases [BBvG]

The analogue of (*) is not true. [NR]

The correction term involves components of the moduli space w/ rational tails (not allowed in quasimap theory) possible the analogue of (*) is true in the quasimap setting.