Mirror symmetry for Painlevé surfaces

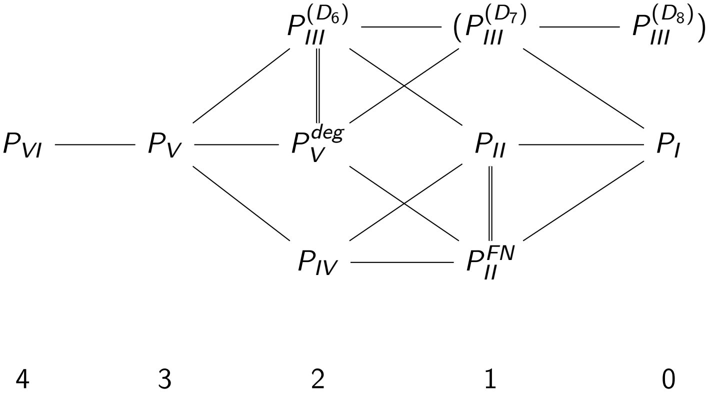
Tom Sutherland

Universidade de Lisboa

28th May 2020

Painlevé equations

The Painlevé equations are second-order differential equations whose only movable singularities are poles



Their solutions provide examples of special functions beyond abelian integrals.

Hamiltonian description

The Painlevé equations have a Hamiltonian description on $\mathbb{C}^2$ with canonical coordinates (q, p)

Hamiltonian description

The Painlevé equations have a Hamiltonian description on $\mathbb{C}^2$ with canonical coordinates (q, p)

The Painlevé I equation $q'' = 6q^2 + t$ has Hamiltonian $H = -p^2 + q^3 + tq$

$$q' = -\frac{\partial H}{\partial p} = 2p \quad p' = \frac{\partial H}{\partial q} = 3q^2 + t$$

After a rational (P_I, P_{II}, P_{IV}), trigonometric (P_{III}, P_V) or elliptic (P_{VI}) transformation of the time variable, all have the form of a particle moving in a time-dependent potential.

Hamiltonian description

The Painlevé equations have a Hamiltonian description on $\mathbb{C}^2$ with canonical coordinates (q, p)

The Painlevé I equation $q'' = 6q^2 + t$ has Hamiltonian $H = -p^2 + q^3 + tq$

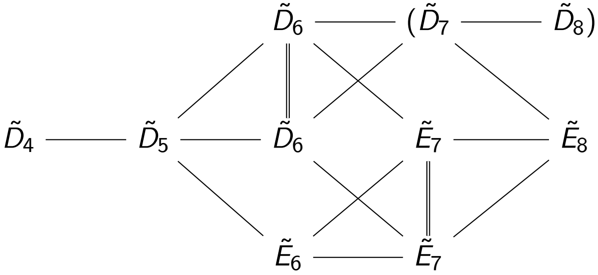
$$q' = -\frac{\partial H}{\partial p} = 2p \quad p' = \frac{\partial H}{\partial q} = 3q^2 + t$$

After a rational (P_I, P_{II}, P_{IV}), trigonometric (P_{III}, P_V) or elliptic (P_{VI}) transformation of the time variable, all have the form of a particle moving in a time-dependent potential.

We need to compactify $\mathbb{C}^2$ to account for solutions with poles.

Spaces of initial conditions

The required compactification is the complement of an anti-canonical divisor of the projective plane blown-up in nine (infinitely near) points.
The canonical holomorphic symplectic form extends to have poles (with multiplicities) on the anti-canonical divisor



The Hamiltonian dynamics of the Painlevé equations can be recovered from (the) locally trivial deformation of the pair.

Limit to rational elliptic surfaces

We can give an autonomous limit of the Painlevé equations by introducing a scaling parameter λ multiplying instances of t in the potential

In the limit $\lambda \to 0$, we find a fast dynamics which limits to a flow around tori.

Limit to rational elliptic surfaces

We can give an autonomous limit of the Painlevé equations by introducing a scaling parameter λ multiplying instances of t in the potential
In the limit $\lambda \to 0$, we find a fast dynamics which limits to a flow around tori.

For example, the limit of Painlevé I is solved by the usual Weierstrass p-function

$$(\mathfrak{p}')^2 = \mathfrak{p}^3 + c\mathfrak{p} + H$$

Limit to rational elliptic surfaces

We can give an autonomous limit of the Painlevé equations by introducing a scaling parameter λ multiplying instances of t in the potential

In the limit $\lambda \to 0$, we find a fast dynamics which limits to a flow around tori.

For example, the limit of Painlevé I is solved by the usual Weierstrass p-function

$$(\mathfrak{p}')^2 = \mathfrak{p}^3 + c\mathfrak{p} + H$$

Geometrically, the space of initial conditions limits to a rational elliptic surface by moving the ninth blow-up point so that all nine points lie on a pencil of cubic curves

Isomonodromy interpretation of Painlevé VI

Fuchs discovered the relationship of Painlevé VI to isomonodromic deformations of connections in a rank 2 bundle over the four-punctured sphere as we vary the complex structure.

Isomonodromy interpretation of Painlevé VI

Fuchs discovered the relationship of Painlevé VI to isomonodromic deformations of connections in a rank 2 bundle over the four-punctured sphere as we vary the complex structure.

For $t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$, Painlevé VI is equivalent to the Schlesinger equations

$$\frac{\partial \mathcal{A}_0}{\partial t} = \frac{[A_0, A_t]}{t} \quad \frac{\partial \mathcal{A}_1}{\partial t} = \frac{[A_1, A_t]}{t-1} \quad \frac{\partial \mathcal{A}_t}{\partial t} = -\left(\frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}\right)$$

for $A_i = A_i(p, q, t)$ 2-by-2 traceless matrices.

Isomonodromy interpretation of Painlevé VI

Fuchs discovered the relationship of Painlevé VI to isomonodromic deformations of connections in a rank 2 bundle over the four-punctured sphere as we vary the complex structure.

For $t \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$, Painlevé VI is equivalent to the Schlesinger equations

$$\frac{\partial \mathcal{A}_0}{\partial t} = \frac{[A_0, A_t]}{t} \quad \frac{\partial \mathcal{A}_1}{\partial t} = \frac{[A_1, A_t]}{t-1} \quad \frac{\partial \mathcal{A}_t}{\partial t} = -\left(\frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}\right)$$

for $A_i = A_i(p, q, t)$ 2-by-2 traceless matrices.

Solutions of the Schlesinger equations define an isomonodromic family of flat connections on the trivial rank 2 bundle on $\mathbb{P}^1$

$$d + \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$$

Isomonodromy for the other Painlevé equations

The remaining Painlevé equations have a similar interpretation as isomonodromic deformations of connections with higher order poles on the projective line.

The monodromy data must be taken to include Stokes data at the higher-order poles, and the complex structure to include local data at these points.

Isomonodromy for the other Painlevé equations

The remaining Painlevé equations have a similar interpretation as isomonodromic deformations of connections with higher order poles on the projective line.
The monodromy data must be taken to include Stokes data at the higher-order poles, and the complex structure to include local data at these points.

Solutions to Painlevé I describe a family of connections on the projective line five Stokes matrices at the unique singular point at ∞ are constant

$$d + \begin{pmatrix} p & z^2 + qz + q + t \\ z - q & -p \end{pmatrix}$$

Character varieties

A moduli space of local systems with simple poles on the four-punctured sphere is a classical character variety

$$\mathrm{Hom}(\pi_1(\mathbb{P}^1 \backslash \{0, 1, t, \infty\}), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2 \cong \mathrm{SL}_2^3//\mathrm{SL}_2$$

Functions are given by traces of holonomies around loops on the curve, and finitely many suffice to generate the coordinate ring.

Character varieties

A moduli space of local systems with simple poles on the four-punctured sphere is a classical character variety

$$\mathsf{Hom}(\pi_1(\mathbb{P}^1\backslash\{0,1,t,\infty\}),\mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2 \cong \mathrm{SL}_2^3//\mathrm{SL}_2$$

Functions are given by traces of holonomies around loops on the curve, and finitely many suffice to generate the coordinate ring.

Similar "wild" character varieties parameterise the monodromy data of connections with higher order poles, e.g. for Painlevé I

$$((\mathbb{P}^1)^5\backslash\Delta)//\mathrm{SL}(2,\mathbb{C})$$

Cubic surfaces

The character varieties are all families of affine cubic surfaces

$$XYZ = f_2(X, Y, Z; a, b, c, \dots)$$

For example, Painlevé VI gives the Fricke-Klein family of cubic surfaces

$$\begin{aligned} XYZ = & X^2 + Y^2 + Z^2 + (ab + cd)X + (ac + bd)Y + (ad + bc)Z \\ & + (a^2 + b^2 + c^2 + d^2 + abcd - 4) \end{aligned}$$

The three coordinate functions X, Y and Z correspond to traces of loops around the three pants curves.
The coefficients a, b, c and d are given by traces around the four simple loops.

Compactification of Painlevé VI

The affine cubic surfaces admit a compactification by a triangle of lines.

The holomorphic symplectic form extends with simple poles to the boundary.

For Painlevé VI, the resulting cubic surface is smooth for generic values of the parameters.

Compactification of Painlevé VI

The affine cubic surfaces admit a compactification by a triangle of lines.
The holomorphic symplectic form extends with simple poles to the boundary.
For Painlevé VI, the resulting cubic surface is smooth for generic values of the parameters.

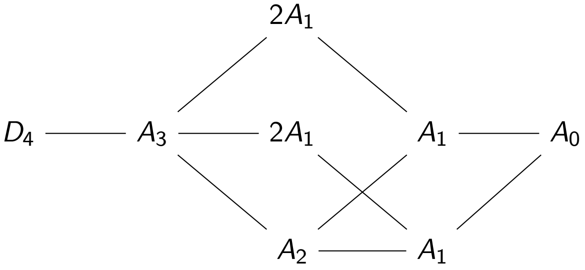
The orthogonal complement to the triangle of lines defines a D_4 sublattice of its Picard lattice.
The 24 lines in the interior represent partially reducible local systems, which correspond to special "truncated" solutions of P_{VI}.

Compactification of the other Painlevé varieties

In the remaining cases, some of the intersections of the triangle of lines at infinity meet at singular points of the cubic surface. These surfaces can be realised by blowing up the projective plane in six points in special position and blowing down effective (-2)-curves.

Compactification of the other Painlevé varieties

In the remaining cases, some of the intersections of the triangle of lines at infinity meet at singular points of the cubic surface. These surfaces can be realised by blowing up the projective plane in six points in special position and blowing down effective (-2)-curves.
The orthogonal complement to these (-2)-classes defines a sublattice of the D_4 lattice.



Non-abelian Hodge and SYZ

The complements of the divisors in rational elliptic surfaces and cubic surfaces are diffeomorphic, but not isomorphic as holomorphic symplectic manifolds.
They belong to a one-dimensional family of holomorphic symplectic manifolds, underlying a hyperkahler structure.

Non-abelian Hodge and SYZ

The complements of the divisors in rational elliptic surfaces and cubic surfaces are diffeomorphic, but not isomorphic as holomorphic symplectic manifolds.
They belong to a one-dimensional family of holomorphic symplectic manifolds, underlying a hyperkahler structure.

Non-abelian Hodge: $\{\text{Higgs bundles}\} \leftrightarrow \{\text{Local systems}\}$

Non-abelian Hodge and SYZ

The complements of the divisors in rational elliptic surfaces and cubic surfaces are diffeomorphic, but not isomorphic as holomorphic symplectic manifolds.
They belong to a one-dimensional family of holomorphic symplectic manifolds, underlying a hyperkahler structure.

Non-abelian Hodge: $\{\text{Higgs bundles}\} \leftrightarrow \{\text{Local systems}\}$

Via hyperkahler rotation, view the elliptic fibration as fibration by special Lagrangian tori.
A scattering diagram is drawn in the base of the elliptic fibration, and the family of cubic surfaces can be constructed via Gross-Siebert.

ADE

We have an ADE-like classification of the possible anticanonical divisors in a rational elliptic surface.

ADE

We have an ADE-like classification of the possible anticanonical divisors in a rational elliptic surface.

Mirror symmetry for the pair of a rational elliptic surface and and anticanonical divisor has been most studied in type A, i.e. a smooth divisor or a cycle of rational curves. A mirror is the pair of a del Pezzo surface together with a smooth anticanonical divisor.

ADE

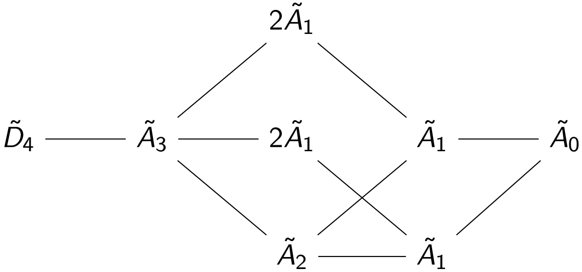
We have an ADE-like classification of the possible anticanonical divisors in a rational elliptic surface.

Mirror symmetry for the pair of a rational elliptic surface and and anticanonical divisor has been most studied in type A, i.e. a smooth divisor or a cycle of rational curves. A mirror is the pair of a del Pezzo surface together with a smooth anticanonical divisor.

Those giving rise to Painlevé equations belong to the D and E series, and from this perspective the mirror pair is a cubic surface with a triangle of lines intersecting in specified singularities.

An identification of lattices

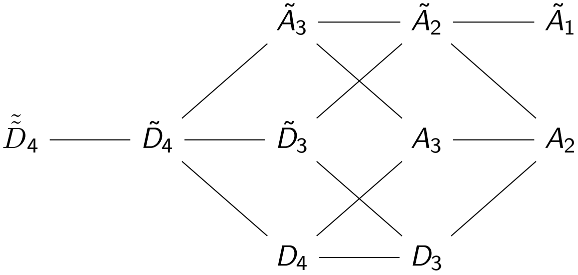
The orthogonal complements to the singular fibres of the Painlevé surfaces are again lattices associated to affine Dynkin diagrams. The further orthogonal complement to a section of the fibration recovers the lattices associated to the affine cubic surfaces.



A lattice polarised mirror symmetry statement similar to type A is expected.

Quivers

We can associate a mutation class of quivers to each of the Painlevé equations.



The mutation equivalence classes each contain a quiver of Dynkin, affine Dynkin or elliptic Dynkin type, in correspondence with the rational, trigonometric and elliptic types of the Painlevé equations.

Stability conditions

The bases of the elliptic fibrations have interpretations as a slice of the space of stability conditions of a Calabi-Yau-3 category associated to the quiver.
The central charge is computed by integrating a meromorphic 1-form along loops in the fibres, whose exterior derivative is the holomorphic symplectic form.

$$Z(S) = \int_{\alpha} \sqrt{z^3 + cz + H} dz$$

Stability conditions

The bases of the elliptic fibrations have interpretations as a slice of the space of stability conditions of a Calabi-Yau-3 category associated to the quiver.

The central charge is computed by integrating a meromorphic 1-form along loops in the fibres, whose exterior derivative is the holomorphic symplectic form.

$$Z(S) = \int_{\alpha} \sqrt{z^3 + cz + H} dz$$

More generally, spaces of quadratic differentials are spaces of stability conditions for associated Calabi-Yau-3 categories.

It is possible to construct the corresponding scattering diagram from the stability conditions perspective.

Cluster varieties

The total spaces of the families of affine cubic surfaces are isomorphic in codimension two to the cluster $\mathcal{X}$-variety of the corresponding quiver.
The scattering diagram can be produced by an iterative process from the data of the quiver.

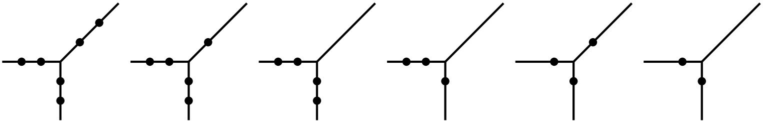
Cluster varieties

The total spaces of the families of affine cubic surfaces are isomorphic in codimension two to the cluster $\mathcal{X}$-variety of the corresponding quiver.
The scattering diagram can be produced by an iterative process from the data of the quiver.

The natural functions on the character varieties can be written as Laurent polynomials in Fock-Goncharov coordinates.
Fock-Goncharov coordinates have an interpretation as holonomies of $\mathbb{C}^*$-local systems on the fibres of the elliptic fibration.

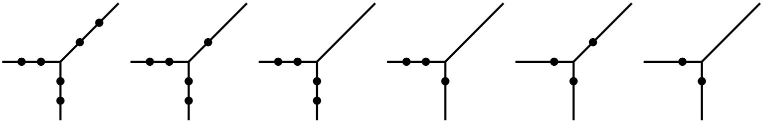
Scattering diagrams

Scattering diagrams for the Painlevé surfaces have (at most) three incoming rays, corresponding to the three lines at infinity.



Scattering diagrams

Scattering diagrams for the Painlevé surfaces have (at most) three incoming rays, corresponding to the three lines at infinity.



Can express the three theta functions as Laurent polynomials via counting broken lines (cf X, Y, Z in Fock-Goncharov coordinates)

Can compute products of theta functions via counts of tropical curves (cf cubic equation satisfied by X, Y, Z)

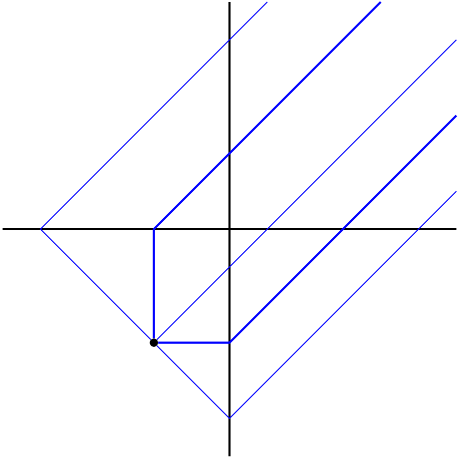
Theta functions for Painlevé VI

In "The mirror of the cubic surface" we find the equation

$$\vartheta_X\vartheta_Y\vartheta_Z = \vartheta_X^2 + \vartheta_Y^2 + \vartheta_Z^2 + (\sum_L z^L)\vartheta_X + (\sum_L z^L)\vartheta_Y + (\sum_L z^L)\vartheta_Z + (\sum_{\alpha\in D_4} z^\alpha - 4)$$

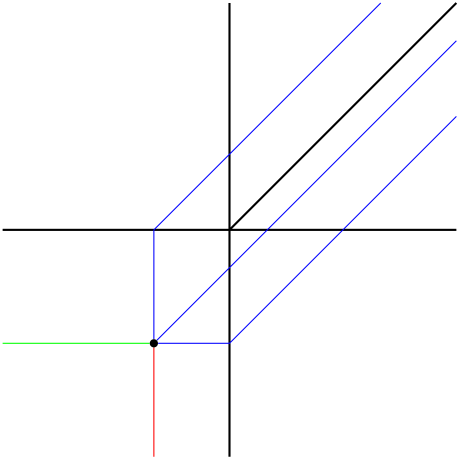
The sums are over lines meeting a given component of the boundary and roots of the D_4 lattice respectively.
After appropriate idenfications, this recovers the Fricke-Klein family.
An analogous result holds for the remaining Painlevé surfaces.

Painlevé VI



$$Z = \frac{y}{x} + \frac{2}{x} + \frac{1}{xy} + \frac{2}{y} + \frac{x}{y}$$

Painlevé I



$$XYZ = X + Y + 1 \qquad X = x \quad Y = y \quad Z = \frac{1 + x + y}{xy}$$

Further Directions

Further Directions

- A uniform treatment

Further Directions

- A uniform treatment
- Homological mirror symmetry

Further Directions

- A uniform treatment
- Homological mirror symmetry
- Schottky uniformisation

Further Directions

- A uniform treatment
- Homological mirror symmetry
- Schottky uniformisation
- Non-abelian theta functions

Further Directions

- A uniform treatment
- Homological mirror symmetry
- Schottky uniformisation
- Non-abelian theta functions
- Abelianisation and broken lines

Further Directions

- A uniform treatment
- Homological mirror symmetry
- Schottky uniformisation
- Non-abelian theta functions
- Abelianisation and broken lines
- Higher dimensions

Further Directions

- A uniform treatment
- Homological mirror symmetry
- Schottky uniformisation
- Non-abelian theta functions
- Abelianisation and broken lines
- Higher dimensions

Further Directions

- A uniform treatment
- Homological mirror symmetry
- Schottky uniformisation
- Non-abelian theta functions
- Abelianisation and broken lines
- Higher dimensions

Thanks!