Sasaki-Einstein metrics on spheres

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§ Introduction

Standard sphere: $S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_n^2 = 1 \}$

manifold

Homotopy sphere: a (real smooth) diff. manifold

homotopy equivalent to S^n

Poincaré conj.

$\Updownarrow$

homeomorphic

Rmk : $\exists$ homotopy spheres NOT diffeomorphic to S^7, these are called exotic spheres

Metrics

(M, g) Riemannian manifold, dim M even

If there exists a complex structure J on M compatible with g, then (M, g) is Kähler.

Ex:
- Rieman surfaces
- projective complex manifold.

(M, g) $\dim M$ odd

Cone $\left(C(M) := M \times \mathbb{R}_{>0} \,,\, \bar{g} = r^2 g + dr^2 \right)$

where r is the coordinate on $\mathbb{R}_{>0}$.

Def. (M, g) is Sasakian if $(C(M), \bar{g})$ admits a Kähler complex structure.

(M, g) is Sasaki-Einstein if $(C(M), \bar{g})$ is Kähler and g is Einstein $(Ric_g = \lambda g)$

Why: String theory

Ex: (S^{2n-1}, g_E) is Sasaki - Einstein

$$(C(S^{2n-1}), \bar{g}) \simeq (\mathbb{C}^n \setminus \{0\}, g_{\mathbb{C}^n})$$

§ Results

Recall: A manifold is parallelisable if its tangent bundle is trivial. S^n bounds a parall. manifold (disk)

Boyer - Galicki - Kollár '05 $\exists$ SE metrics on any homotopy sphere Σ^{4n+1} $(n \geq 1)$ that bounds parall. manifolds.

Conj: True also in dim $4n-1$.

Collins - Székelyhidi '19 $\exists$ ∞-many SE metrics on S^5.

Conj: $\exists$ ∞-many SE metrics on every standard S^{2n-1}, $n \geq 3$.

Thm (Liu - Sano - T.) Any homotopy sphere Σ^{2n-1} that bounds parall. manifolds admits ∞-many SE metrics.

§ The construction

$n \geq 3$, $a = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$, $a_i > 1$.

$$Y(a) := \{ z_0^{a_0} + z_1^{a_1} + \dots + z_n^{a_n} = 0 \} \subseteq \mathbb{C}^{n+1}$$

Brieskorn-Pham sing.

Milnor '68: The link $L(a) := Y(a) \cap S^{2n+1}$ is a smooth compact simply connected $(2n-1)$-manifold that bounds a parall. manifold.

Thm A (-) Assume $a_0 \leq a_1 \leq \dots \leq a_n$. $L(a)$ admits a SE-metric iff $1 < \sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{a_n}$.

"Proof"

$d := \mathrm{lcm}\{a_i\}$, $d_i = \frac{d}{a_i}$

$$\begin{array}{ccc} \mathbb{C}^* \circlearrowright Y(a) & \subseteq & \mathbb{C}^n \\ \text{quot.} \downarrow & & \downarrow \\ X^{orb} & \subseteq & \mathbb{P}(d_0, \dots, d_n) \end{array}$$

$\lambda \cdot (z_0, \dots, z_n) = (\lambda^{d_0} z_0, \dots, \lambda^{d_n} z_n)$

weighted hyp.

Boyer - Galicki: $L(a)$ admits a SE metric iff

X^{orb} admits a Fano KE-metric.

Note: X^{orb} Fano $\Leftrightarrow$ $-K_{X^{orb}}$ is ample $\Leftrightarrow$ $1 - \sum \frac{1}{a_i} < 0$.

Roughly speaking X^{orb} admits a KE-metric iff X^{orb} is K-polystable (deep)

It's easy to show that X^{orb} is K-polystable. $\square$

§ Explicit examples $n \geq 3$

finite ab group

Kervaire - Milnor

$$\Theta_{2n-1} = \left\{ \begin{array}{c} \text{homotopy spheres} \\ \dim = 2n-1 \end{array} \right\} \Big/ \begin{array}{l} \text{oriented} \\ \text{diff} \end{array}$$

$$\vee$$

$$bP_{2n} = \begin{array}{c} \text{spheres that} \\ \text{bound parall manifolds} \end{array}$$

subgroup

$n = 2m+1$ odd $\qquad bP_{4m+2}$ is either 0 or $\mathbb{Z}_2$

$n = 2m$ even $\qquad bP_{4m}$ is <u>big</u>

$Y(a) = \{ z_0^{a_0} + \dots + z_n^{a_n} = 0 \}$ $\quad L(a)$ link

graph $G(a)$: vertices a_i

a_i and a_j are connected if $\gcd(a_i, a_j) \neq 1$.

Brieskorn: if $G(a)$ contains at least two isolated points,

then $L(a)$ is hom. to S^{2n-1}.

Assume $L(a) \in bP_{4m}$. The diffeomorphism type of $L(a)$

is given by $\dfrac{\tau(a)}{8} \mod |bP_{4m}|$

where τ is the signature of the Milnor fibre.

Note: there is a combinatorial formula for τ.

Brieskorn spheres: $a = (2, 2, \ldots, 2, 3, 6k-1) \in \mathbb{Z}^{n+1}$

$n = 2m$

$$L(a) \in bP_{4m} \quad , \quad \frac{\tau(a)}{8} = (-1)\, k \mod |bP_{4m}|$$

so all exotic spheres in bP_{4m} are constructed.

We generalised such examples.