Forbidden patterns in tropical planar curves and panoptigons

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Online Nottingham algebraic geometry seminar

Outline

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Big face graphs

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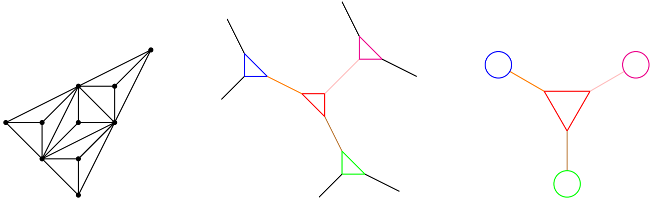
Introduction

A tropically planar graph is a metric graph, G which is dual to a regular unimodular triangulation Δ of some lattice polygon P. Here we are concerned with the question: Which graphs G occur in this way? We refer to graphs which do occur in this way as realizable graphs (also called troplanar graphs).



Duality

$$\text{Tropical curve } C \longleftrightarrow \text{Regular subdivision of Newton polygon } P$$

$$\text{Smooth tropical curve} \longleftrightarrow \text{Unimodular triangulation}$$
$$\text{Genus} = \# \text{ of cycles in } C \longleftrightarrow \# \text{ of interior lattice points in } P$$

Each curve C contains a underlying metric graph of genus g which is called the *skeleton* of C, and it is a planar graph with g distinguished cycles.

In 2015 Brodsky et al. studied the moduli space of tropical plane curves of genus g, for g = 3, 4 and 5, by analyzing the associated moduli space of metric graphs.

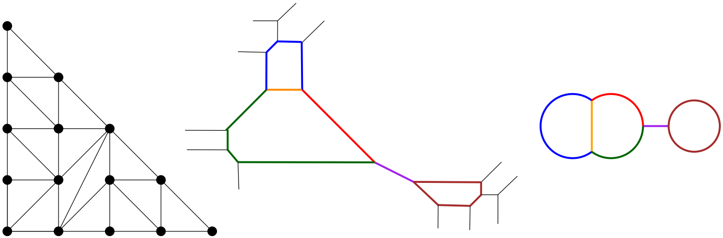


Figure 1: Unimodular triangulation, tropical quartic curve, and the skeleton

Moving out edges of a polygon

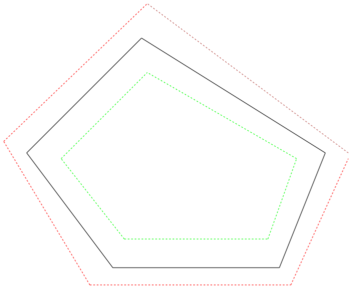


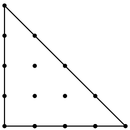
Figure 2: Moving out edges of a polygon

Theorem (Koelmann 1991)

Let $\Delta \subset \mathbb{R}^2$ be a two dimensional polygon, such that $\Delta^{(1)}$ is again two-dimensional. Then $\Delta^{(1)(-1)}$ is a lattice polygon containing Δ.

We call a lattice polygon Δ *maximal* if $\Delta = \Delta^{(1)(-1)}$.

$g = 3$



T_4 has 1278 regular triangulations.

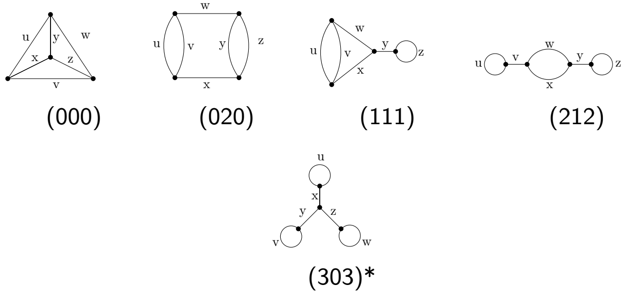
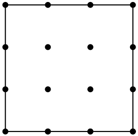
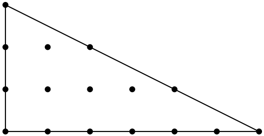


Figure 3: The five trivalent graphs of genus 3

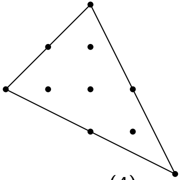
$g = 4$



$Q_1^{(4)} = R_{3,3}$



$Q_2^{(4)}$

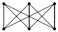


$Q_3^{(4)}$

$Q_i^{(4)}$	No. of Triangulations
$Q_1^{(4)}$	5941
$Q_2^{(4)}$	1278
$Q_3^{(4)}$	20

$g = 4$


(000)A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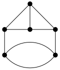
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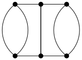
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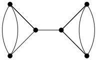
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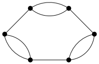
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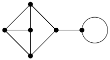
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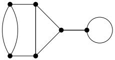
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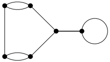
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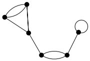
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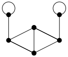
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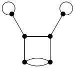
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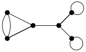
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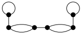
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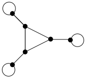
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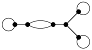
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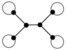
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Troplanarity is not minor closed

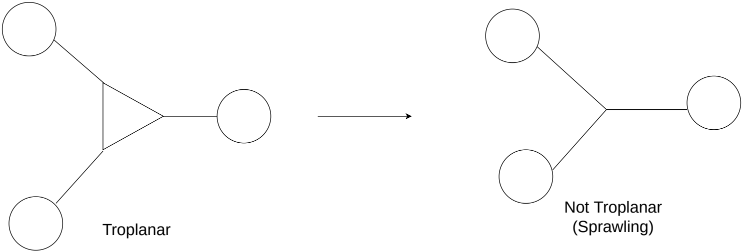

Figure 5: Genus four graph is realizable but genus three minor is sprawling

Prior Known Criteria

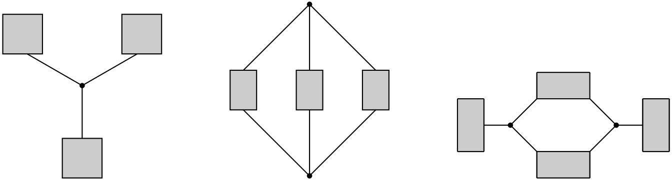

Figure 6: Graph with a sprawling node (left), a crowded graph (center), and a TIE-fighter graph (right).

Splits

A *split* is a subdivision with exactly two maximal cells; it is necessarily regular.
Cut edges in the skeleton correspond to *splits* in the unimodular triangulation.
Two splits are *compatible* if their split lines do not meet in the interior of P.

Lemma

Splits corresponding to distinct cut edges are compatible.

Heavy Cycle

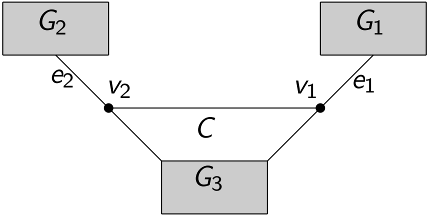

Figure 7: Graph with the heavy cycle C

P' is the subpolygon which realizes $G_3 \cup C$.

Lemma (Joswig,T '20)

Suppose that G has a heavy cycle with cut edges e_1 and e_2 as in Figure 7. Then the triangles T_1 and T_2 in Δ share an edge $[z, w]$, where z is the interior lattice point dual to C, and the split lines S_1 and S_2 intersect in w, which is a vertex of P', and which lies in the boundary of P.

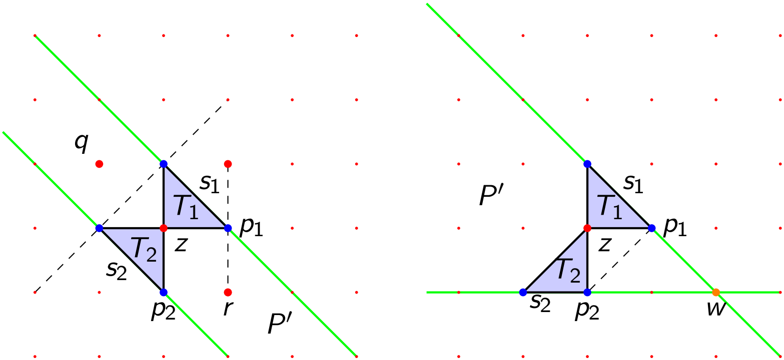

Figure 8: Two possibilities for S_1 and S_2, which are ruled out a posteriori in the proof. Left: S_1 and S_2 are parallel. Right: S_1 and S_2 intersect at a point.

Sprawling Triangle

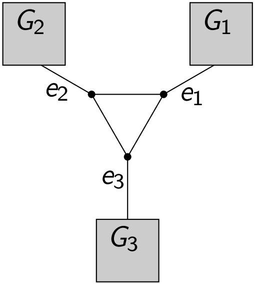

Figure 9: A graph with a sprawling triangle

Theorem (Joswig,T '20)

If G has a sprawling triangle then $g = 4$, and, up to unimodular equivalence, we have

$$P = \operatorname{conv}((-2,0),(0,-2),(2,2)) \quad \text{and}$$
$$\Delta = \text{anti-honeycomb triangulation of genus four.}$$

Heavy Cycle with Two Loops

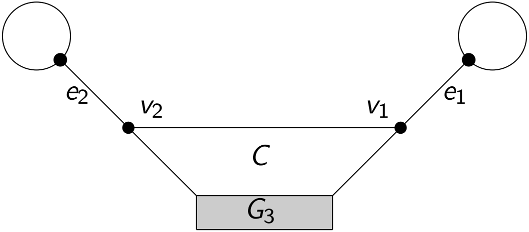

Figure 10: Heavy cycle with two loops

Theorem (Joswig,T '20)

Suppose G is a graph with a heavy cycle C and two loops with cut edges, e_1 and e_2, as in Figure 10. Then the heavy component P' can have at most three interior lattice points, and these lie on the line spanned by the edge $[z, w] \in \Delta$, where z is the interior lattice point dual to C, and w is the intersection point of the split edges s_1 and s_2. In particular, P' is hyperelliptic and $g \leq 5$.

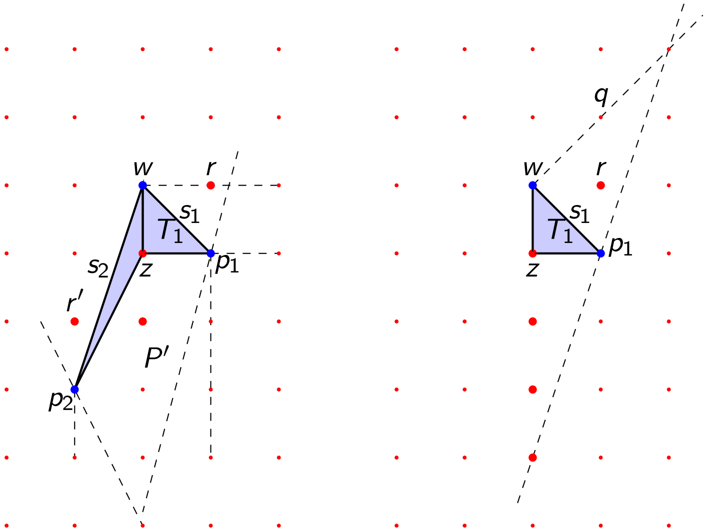

Figure 11: This illustrates the previous theorem: general sketch (left) and the case when $g(P') \geq 4$ (right), which is impossible

Theorem (Joswig,T '20)

A trivalent planar graph of genus $g \leq 5$ is not tropically planar if one of the following holds:

- it contains a sprawling node, or
- it contains a sprawling triangle and $g \geq 5$, or
- it is crowded, or
- it is a TIE-fighter, or
- it has a heavy cycle with two loops such that the interior lattice points of the heavy component do not align with the intersection of the two split lines.

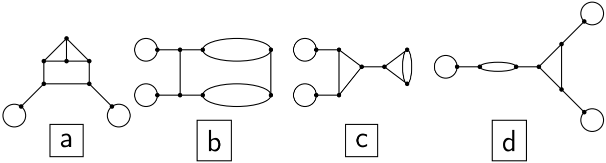

Figure 12: The four genus 5 graphs that are ruled out uniquely by our criteria

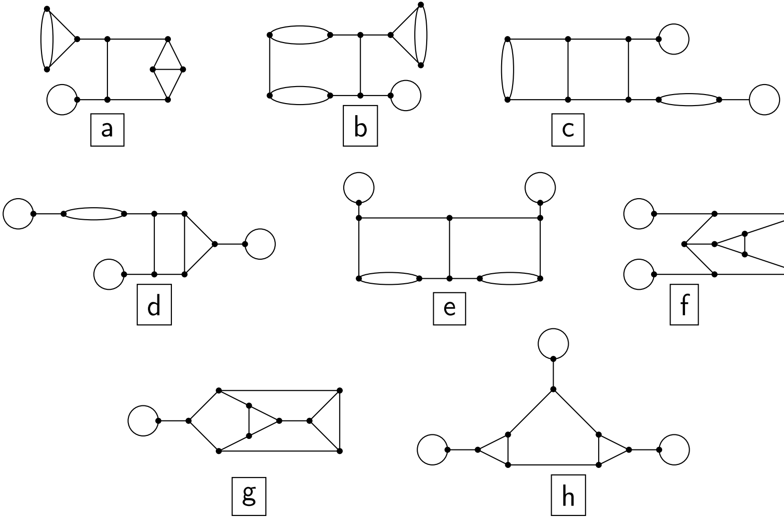

Figure 13: The eight trivalent planar graphs of genus 6, which are not tropically planar, and not ruled by any known criteria up till 2020.

Heavy cycle with one loop

Theorem (T' 22)

Suppose G is a tropically planar graph with a heavy cycle with one loop as shown in Figure 14, then the heavy component P_3 is hyperelliptic and can have at most three interior lattice points. Also, P_2 can have at most three interior lattice points. In the case when genus $g = 6$ and $g(P_2) = 2$, P_2 is hyperelliptic and the triangulation restricted to P_2, i.e, Δ_2 cannot have a nontrivial split. In particular, genus of G can be at most seven.

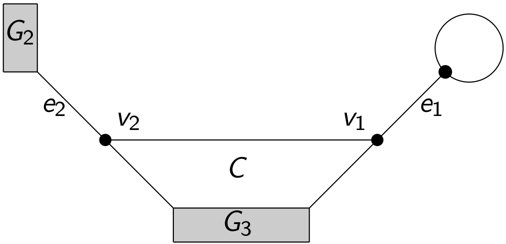

Figure 14: Heavy cycle with one loop

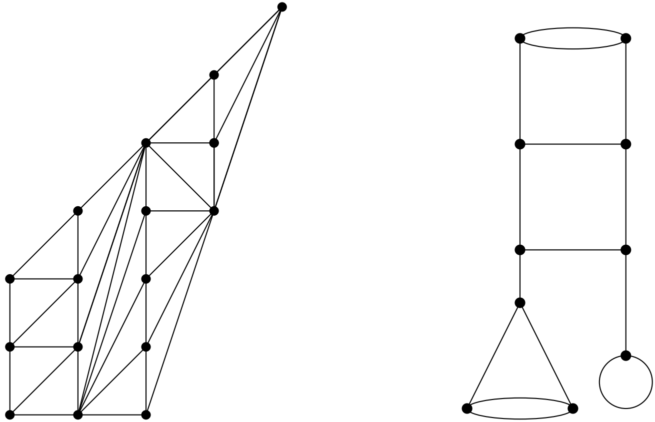

Figure 15: A unimodular triangulation of genus six with $g(P_2) = 2$ (left), corresponding skeleton with a heavy cycle with one loop with G_2 that does not have a cut edge (right)

Double Heavy cycle

Lemma (T' 22)

Suppose that G has double heavy cycles C_1 and C_2 with cut edges e_1 and e_2 as in Figure 7. Then the triangles T and T_1 in Δ share an edge $[z_1, w]$, the triangles T and T_2 in Δ share an edge $[z_2, w]$ where z_1 is the interior lattice point dual to C_1 and z_2 is the interior lattice point dual to C_2. The split lines S_1 and S_2 intersect in w, which is a shared vertex between T_1 and T_2.

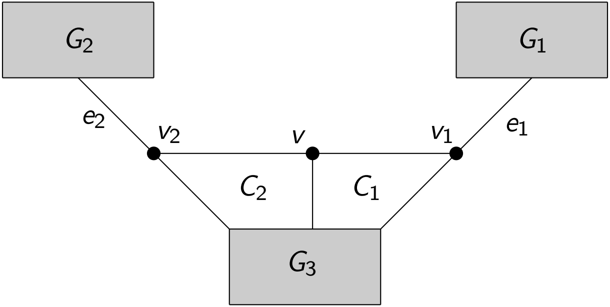

Figure 16: Graph with double heavy cycles C_1 and C_2

Double Heavy Cycle with two loops

Theorem (T ' 22)

Suppose G is a graph with double heavy cycles C_1 and C_2 with two loops with cut edges, e_1 and e_2, as in Figure 17. If $g(P) = 6$, then the interior lattice polygon of the heavy component, i.e., $int(P_3)$ is a unit parallelogram.

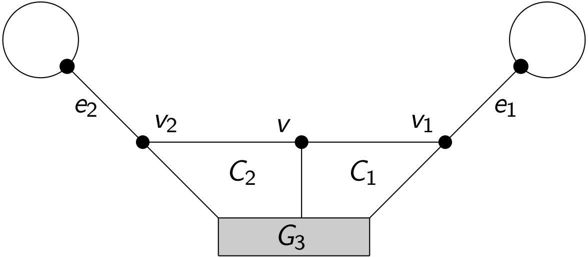

Figure 17: Double heavy cycle with two loops

Theorem (T '22)

Obstructions to a troplanar graph $G \neq enve - loop$ $(g \geq 6)$,

- *G contains a sprawling node, or G contains a sprawling triangle and $g \geq 5$, or*
- *G is crowded, or G is a TIE-fighter, or*
- *G has a heavy cycle with two loops such that the interior lattice points of the heavy component do not align with the intersection of the two split lines, or*
- *the components of G after bridge reduction are not tropically planar, or*
- *G has a heavy cycle with one loop such that either the interior lattice points of the heavy component do not align with the intersection of the two split lines or the connected component with genus greater than one has a cut edge, or*
- *G has a double heavy cycle with two loops such that either the interior lattice points of the heavy component do not form a unit $||^{gm}$ or no three cycles in the heavy component share a vertex.*

Anti-Honeycomb

$$L_k = \{y{=}2x{+}k\}\,,\;\; M_\ell = \{y{=}x/2{-}\ell/2\}\,,\;\; N_m = \{y{=}{-}x{+}m\}\;\;,$$

$$A_{(0,k;0,k;0,k)} \;=\; \mathrm{conv}\{(-k,-k),(0,k),(k,0)\}$$

$$g(A_{(0,k;0,k;0,k)}) \;=\; \frac{3k^2-3k+2}{2}\;\;.$$

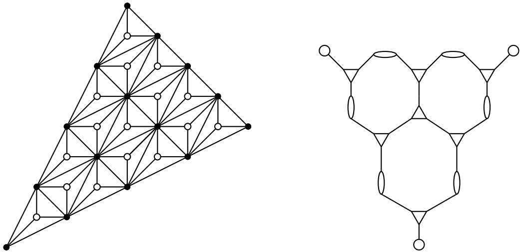

Figure 18: Anti-honeycomb triangulation of genus 19 on the left, and the corresponding skeleton on the right

Panoptigons

Definition

A convex lattice polygon P is a *panoptigon* if P contains a lattice point p such that all other lattice points in P are visible from p. We call p a *panoptigon point* for P.

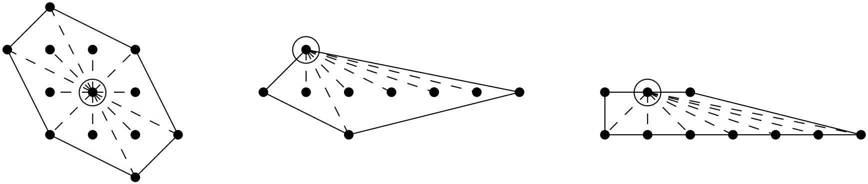

Figure 19: Three panoptigons, with a panoptigon point circled and lines of sight illustrated.

Panoptigons with $\mathrm{lw}(P) \geq 3$

Theorem (Morrison,T '20)

Let P be a panoptigon with lattice width $\mathrm{lw}(P) \geq 3$. Then $|P \cap \mathbb{Z}^2| \leq 13$.

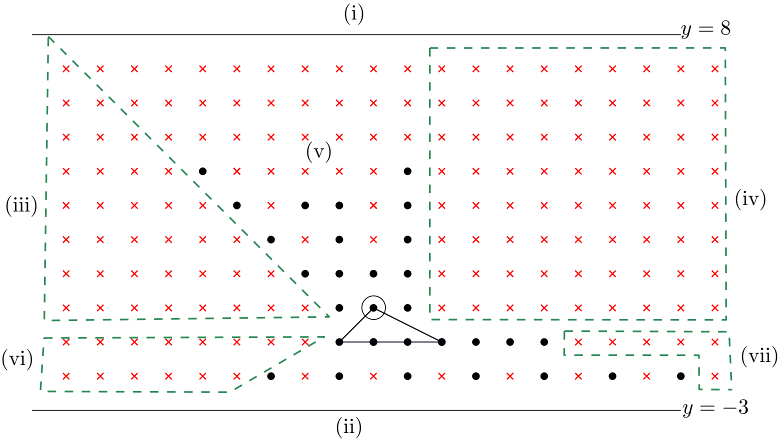

Figure 20: Possible lattice points in P.

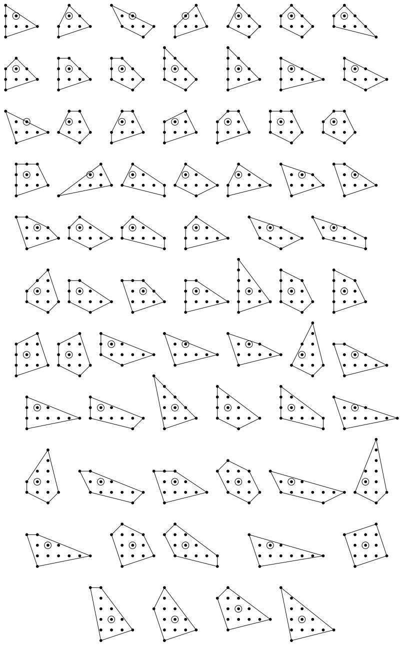

Figure 21: Nonhyperelliptic panoptigons with lattice diameter at least 3

Panoptigons with $l(P) \leq 2$

Proposition

Let P be a nonhyperelliptic panoptigon of lattice diameter at most 2. Then up to lattice equivalence P is either the triangle $\mathrm{conv}((0,1),(0,3),(4,0))$, the quadrilateral $\mathrm{conv}((1,0),(2,0),(3,1),(0,3))$, or the quadrilateral $\mathrm{conv}((0,1),(0,2),(2,3),(3,0))$.

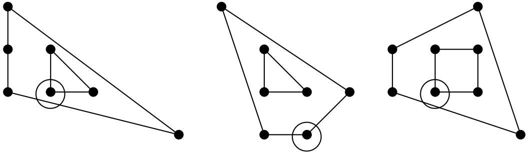

Figure 22: The three nonhyperelliptic panoptigons with $l(P) \leq 2$.

Corollary

Up to lattice equivalence, there are 72 nonhyperelliptic panoptigons.

Big face Graphs

Definition

A trivalent planar graph G is said to be a *big face graph* if for any planar embedding of G, there exists a bounded face that shares an edge with every other bounded face.

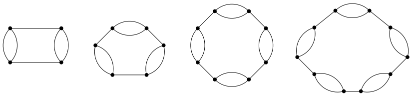

Figure 23: The loop of loops L_g for $3 \leq g \leq 6$

Hence, P_{int} for a realizable big face graph needs to be a panoptigon.

Theorem (Morrison,T)

Let G be a big face graph of genus $g \geq 14$. The graph G is not tropically planar.

This bound can be made even better by checking the list of panoptigons.

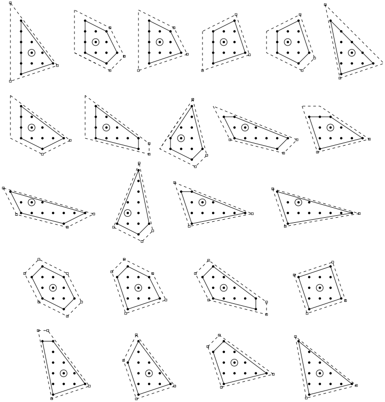

Figure 24: All nonhyperelliptic panoptigons with 12 or 13 lattice points, along with their relaxed polygons

Future problems

- Panoptitopes (with Gennadiy Averkov and Ralph Morrison)
  If P is a panoptitope and L is a half integral polytope, such that $2L = P$, then
  - if the panoptigon point is on the boundary, then L is hollow.
  - if the panoptigon point is in the interior, then L has a unique interior point.

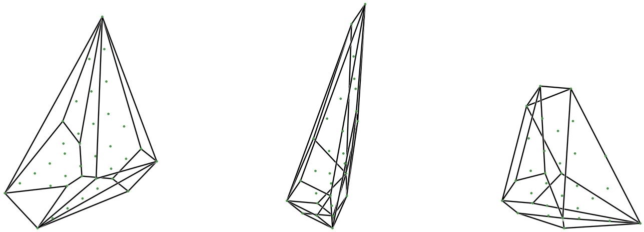

Figure 25: Examples of maximal panoptitopes

Chessboards and Lattice Visibility

- Line of attack in chess is the same as a line of visibility, i.e an attack piece should see the attacked piece in via a straight line (except the knight).
- What is the maximum number of pieces which could be attacked in a single move, if all lines of sight are allowed as lines of attack on a convex lattice chessboard?

- Generalization of moving out for polytopes in three dimensions, and connection to realizability of tropical surfaces.
- Understand locus of tropical plane curves in $\mathbb{M}_g$ by using forbidden criteria analysis.
- Extending the computation of skeletons for non-smooth tropical curves.

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