A relative Yau-Tian-Donaldson Conjecture and Stability Thresholds

University of Nottingham
Algebraic Geometry Seminar
18/05/2023

Setting: X^m smooth Fano manifold, $Aut(X)^0=\{Id\}$, $\omega\in c_1(X)$ Kähler form

1 Introduction & Motivation

Theorem (Chen-Donaldson-Sun '15, Berman '16, Fujita-Odaka '18, Blum-Jonsson '21, Liu-Xu-Zhuand '22)

The followings are equivalent:
i) X admits a Kähler-Einstein metric;
ii) $(X,-K_X)$ is (uniformly) K-stable;
iii) $\delta>1$

It solves the famous Yau-Tian-Donaldson Conjecture on Fano manifolds and it produces a fascinating link between Differential and Algebraic Geometry

1.1 Delta-invariant

Def $\delta=\inf_{E/X}\frac{A_X(E)}{S(E)}$ where
- the infinum is over all prime divisors E of $Y\xrightarrow{\rho_Y}X$, ρ_Y birational morphism ($Y\overset{\rho_Y}{\geq}X$);
- $A_X(E):=1+ord_E(K_{Y/X})$ is the log discrepancy;
- $S(E):=\frac{1}{(-K_X^n)}\int_0^{+\infty}vol_Y(\rho_Y^*(-K_X)-tE)dt$ is the expected vanishing order.

Rmk:
- $\delta=\lim_{m\to\infty}\delta_m:=\lim_{m\to+\infty}\inf\left\{lct(D):D=\frac{1}{mN_m}\sum_{s=0}^{N_m}(s_s=0),\ s_s \text{ BASIS OF } H^0(-mK_X),N_m=h^0(-mK_X)\right\}$
- $\delta>1$ is an efficient criterion to detect K-stability (Abban-Zhuang '22).

Ex: $X=\mathbb{P}^2$, $Z=\{p_1,p_2,p_3,p_4\}$, $Y=Bl_ZX$.
$\psi\in C^\infty(X\setminus Z)$ SUCH THAT $\psi=_{loc}\log(\|z\|^2)$ AROUND p_i
$\Rightarrow$ "$[\psi]$-KE METRIC $=$ KE METRIC ON Y"

1.2 Kähler-Einstein metrics

Def A **Kähler-Einstein metric** is determined by a Kähler form $\omega_{KE}\in c_1(X)$ such that

$$Ric(\omega_{KE})=\omega_{KE}$$

- Locally $Ric(\omega)=_{loc}-\frac{i}{\pi}\partial\bar\partial\log(\det g_{j\bar k})$ IF $\omega=i\sum g_{j\bar k}dz_j\wedge d\bar z_k$
- $\omega_{KE}=\omega+\frac{i}{\pi}\partial\bar\partial\varphi$ for φ solution to the complex Monge-Ampère equation

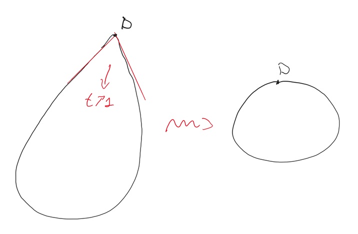
$$\begin{cases}(\omega+\frac{i}{2\pi}\partial\bar\partial\varphi)^m=e^{-\varphi}dV\\ \varphi\in\mathcal{H}_\omega\end{cases}$$

where dV is a smooth volume form attached to (X,ω), while $\mathcal{H}_\omega:=\{u\in C^\infty(X):\omega+\frac{i}{\pi}\partial\bar\partial u \text{ is Kähler}\}$

1.3 A small disgression

"Idea behind CDS '15" (from K-stability to KE metrics): a variant of classical continuity methods

- Fix $D\in|-mK|$ smooth.
- Seek for "log KE metrics", i.e. $\omega_t=\omega+\frac{i}{\pi}\partial\bar\partial u_t$ where:
 i) $u_t\in C^\infty(X\setminus D)$ solves $\begin{cases}Ric(\omega_t)=t\omega_t\\ \omega_t>0\end{cases}$ over $X\setminus D$
 ii) $u_t\in L^\infty(X)$ with cone singularities along $D_t:=\frac{1-t}{m}D$



$\Rightarrow$ Globally $\begin{cases}(\omega+\frac{i}{\pi}\partial\bar\partial u_t)^m=e^{-tu_t-\frac{1-t}{m}\log|s_D|_h^2}dV\\ u_t\in L^\infty(X)\cap\underbrace{PSH(X,\omega)}\end{cases}$

singular metrics on $-K$ ($\omega+\frac{i}{\pi}\partial\bar\partial u_t\geq 0$ in a weak sense)

$\Rightarrow$ defining $V_t:=tu_t+\frac{1-t}{m}\log|s_D|_h^2\in PSH(X,\omega)$ we have

$\begin{cases}(\omega+\frac{i}{\pi}\partial\bar\partial V_t)^m=e^{-V_t}dV\\ V_t\in PSH(X,\omega),\ \|V_t-\underbrace{\frac{1-t}{m}\log|s_D|_h^2}\|_\infty\leq C\end{cases}$ $\Longleftarrow$ $Ric(\omega+\frac{i}{\pi}\partial\bar\partial V_t)=\omega+\frac{i}{\pi}\partial\bar\partial V_t$ $\Rightarrow$ **KE metrics with prescribed singularities**

prescribed singularities

In general: Let (Y,B_Y) be a "log Fano pair" ($-K_Y-B_Y$ AMPLE) such that
- $Y\overset{\rho_Y}{\geq}X$
- $\rho_{Y,*}(-K_Y-B_Y)=-K_X$

$\Big\}\ (Y,B_Y)\overset{\rho_Y}{\geq}X$

$\left\{\begin{matrix}\text{Log KE metrics}\\ \text{on } (Y,B_Y)\end{matrix}\right\}\subset\left\{\begin{matrix}\text{KE metrics with}\\ \text{prescribed singularities on X}\end{matrix}\right\}$

"$(Y,B_Y)\longrightarrow(X,\psi_{(Y,B_Y)})$" $\in PSH(X,\omega)$, the "prescribed singularities"

Main Advantages:
- the geometry of X is simpler;
- we can compare the existence of log KE metrics (hence the "log K-stability notions") above X working in the same cohomology class $c_1(X)$.

1.4 Natural questions?

- Is there a natural set of prescribed singularities that can be read algebraically?
- If yes, is there a natural "K-stability notion" for these algebraic prescribed singularities?
- What about the "Yau-Tian-Donaldson conjecture" in this setting?
- Is there a valuative criterion (delta-invariant)?
- How K-stability behaves varying the prescribed singularities?

2 D-log K-stability

From now on we assume $L=-K_X$

Recall: A (Weil) b-divisor $D=\{D_Y\}_{Y\geq X}$ is a family of $\mathbb{R}$-divisors such that $\rho_*D_{Y'}=D_Y$ if $Y'\overset{\rho}{\geq}Y$

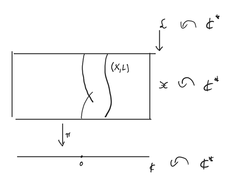
RMk:
- Let D be a divisor on X. Then $D_Y:=\rho_Y^*D$ is a b-divisor.
- A b-divisor D is said to be "Cartier" if there exists $Y\geq X$ such that $D_{Y'}=\rho^*D_Y$ for any $Y'\overset{\rho}{\geq}Y$

Def: We define $Div_L(X)$ as the set of all generalised b-divisors (countable sum is allowed) such that
i) L-D is Nef;
ii) $\langle(L-D)^m\rangle:=\inf_{Y\geq X}vol_Y(\rho_Y^*L-D_Y)>0$

Proposition: There exists a natural set of algebraic singularities $\mathcal{M}^{ALG}\subseteq PSH(X,\omega)$ such that $Div(X)\overset{1:1}{\longleftrightarrow}\mathcal{M}^{ALG}$

2.1 Usual K-stability

Def: A test configuration $(\mathcal{X},\mathcal{L})$ for (X,L) consists of
i) a normal variety $\mathcal{X}$;
ii) a flat and projective morphism $\pi:\mathcal{X}\to\mathbb{C}$;
iii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}^*$;
iv) a $\mathbb{C}^*$-linearized Q-line bundle $\mathcal{L}$;
v) an isomorphism $(\mathcal{X}_1,\mathcal{L}_1)\simeq(X,L)$.



$|-mK_X|$
$X\hookrightarrow\mathbb{P}^N$
1-ps in $\mathbb{P}^N$
$\Rightarrow\mathcal{X}$

Def: (X,L) is uniformly K-stable if there exists A>0 such that $DF(\mathcal{X},\mathcal{L})\geq AJ(\mathcal{X},\mathcal{L})$ for any $(\mathcal{X},\mathcal{L})$ ample test configuration where

The "Donaldson-Futaki" invariant is defined as $DF(\mathcal{X},\mathcal{L}):=\frac{1}{(L^m)}(K_{\bar{\mathcal{X}}/\mathbb{P}^1}\cdot\bar{\mathcal{L}}^m)+\frac{m}{(L^m)}\frac{(\bar{\mathcal{L}}^{m+1})}{m+1}$
where we considered the compactification $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ over $\mathbb{P}^1$.

The J-functional is a "measures" on the triviality of test configuration and it vanishes is and only if $\mathcal{X}\simeq X\times\mathbb{C}$

2.2 A Riemann-Zariski perspective

We consider $X^{RZ}:=\varprojlim_{\mathcal{X}}\mathcal{X}$, $\mathcal{X}\overset{\mathbb{C}^*}{\geq}X\times\mathbb{P}^1$. We then have $N^p(X^{RZ}):=\varprojlim_{\mathcal{X}}N^p(\mathcal{X})$ p-codimensional Weil classes (p-codimensional classes)
and $CN^p(X^{RZ}):=\varinjlim_{\mathcal{X}}N^p(\mathcal{X})$ p-codimensional Cartier classes.

$\Rightarrow N^1(\mathcal{X})\longrightarrow CN^1(X^{RZ})\longrightarrow N^1(X^{RZ})$
$c_1(\mathcal{L})\longmapsto$ "$\mathcal{L}$" (by abuse of notation)

i.e. well-defined "(ample) test configuration classes"

Def: Let $D\in Div_L(X)$. We define $\mathcal{D}\in N^1(X^{RZ})$ as the class of the following b-divisor.
Let $\mathcal{X}\overset{\mathbb{C}^*}{\geq}X\times\mathbb{P}^1$, $\mathcal{X}_{|\pi^{-1}(\mathbb{C}^*)}\simeq Y\times\mathbb{C}^*$ for $Y\geq X$. If $D_Y=\sum_F a_F F$, then $\mathcal{D}_{\mathcal{X}}:=\sum a_F\mathcal{F}_{\mathcal{X}}$ where $\mathcal{F}_{\mathcal{X}}$ is the Zariski closure of $F\times\mathbb{C}^*$ with respect to the open embedding $Y\times\mathbb{C}^*\hookrightarrow\mathcal{X}$

$\mathcal{X}$
$D\times\mathbb{C}^*\subseteq Y\times\mathbb{C}^*\hookrightarrow Y\times\mathbb{P}^1\longrightarrow X\times\mathbb{P}^1$

Main idea: The "positivity" to take under considerations to define "D-log K-stability" are encoded in the classes $\mathcal{L}-\mathcal{D}\in N^1(X^{RZ})$, where $\mathcal{L}$ varies among all ample test configuration classes

Proposition/Definition:
There exists a well-defined Donaldson-Futaki invariant
$\{\text{test configuration classes}\}\longrightarrow\mathbb{R}$
$\mathcal{L}\longmapsto DF(\mathcal{L};D)$
such that
i) it coincides with the usual Donaldson-Futaki invariant when D=0;
ii) it is given in terms of an intersection formula in X^{RZ}.
Moreover
$$DF(\mathcal{L};D)=\sup_{Y'}\inf_{Y\geq Y'}DF_{B_Y}(\mathcal{Y},\mathcal{L}_{m,\mathcal{Y}})$$
"Asymptotic" log K-stability
for $B_Y=D_Y-K_{Y/X}$, $L_Y:=\rho_Y^*L-D_Y$, while $\mathcal{L}_{m,\mathcal{Y}}:=\tilde\rho_Y^*\mathcal{L}+m\mathcal{Y}_0-\mathcal{D}_{\mathcal{Y}}$ w.r.t.

$$\begin{array}{ccc}\mathcal{Y}&\xrightarrow{\tilde\rho_Y}&\mathcal{X}\\ \downarrow&&\downarrow\\ Y\times\mathbb{P}^1&\xrightarrow{\rho_Y\times Id}&X\times\mathbb{P}^1\end{array}$$

Def: (X,L) is said to be uniformly D-log K-stable if there exists A>0 such that $DF(\mathcal{L};D)\geq A\,J(\mathcal{L};D)$ for any ample test configuration class $\mathcal{L}$.

It measures the "D-triviality".

3. Main Results

"Analytic" "Algebraic"

Theorem
Let $L=-K_X$, $D\in Div_L(X)$ (i.e. $\psi_D\in\mathcal{M}^{ALG}$).

There exists $[\psi_D]$-KE metric $\longrightarrow$ (X,L) is uniformly D-Log K-stable ③
① $\updownarrow$ ② $\uparrow$ ④
The ψ_D-Ding functional is "coercive" $\longleftrightarrow$ (X,L) is uniformly D-log Ding-stable
$\updownarrow$
$\delta_D>1$ ⑤
$\updownarrow$
$\tilde\delta_D>1$ ⑥

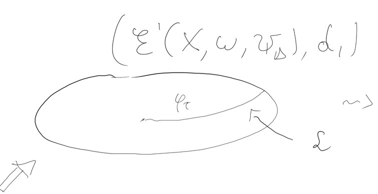
Main ideas of the proofs:

1<->2 : It was part of my PhD thesis. The idea is to study (weak) solution to the associated complex Monge-Ampère equation thanks to a variational approach in pluripotential theory. A new "strong metric topology" plays a key role.

2->4 : To any ample test configuration class we can associate a natural geodesic ray in the metric space of the previous point. Then a singular version of Deligne-Pairings allows us to connect the slope of the Ding functional to the "D-log Non-Archimedean Ding functional", which measures the D-log Ding-stability. Many difficulties coming from the singularities and from adapting pluripotential analysis to the Riemann-Zariski perspective.

4->3 : Use the "log asymptotic" formula and known results/strategies in the literature.

$(\mathcal{E}^1(X,\omega,\psi_D),d_1)$



$\rightsquigarrow\lim_{t\to+\infty}\frac{D(\varphi_t)}{t}=D(\mathcal{L};D)$

3.2 Stability thresholds

For simplicity let assume that D is Cartier.
Namely $D=D_Y$ is a divisor on $Y\overset{\rho}{\geq}X$ such that $L_Y:=\rho^*L-D_Y$ is nef and big. Then (Y,B_Y) is a (weak) log Fano pair, where $B_Y:=D_Y-K_{Y/X}$

Then $\delta_D:=\inf_{E/X}\frac{A_X(E)-ord_E(D)}{S_D(E)}=\underbrace{\inf_{E/Y}\frac{A_{(Y,B_Y)}(E)}{S^{L_Y}(E)}}_{=\delta_{(Y,B_Y)}\text{ usual "log delta-invariant"}}$

$E\subset Z\overset{\rho_Z}{\geq}Y$
$S^{L_Y}(E)=\frac{1}{(L_Y^m)}\int_0^{+\infty}vol_Z(\rho_Z^*L_Y-tE)dt$

$\tilde\delta_D:=\inf_{E/X}\frac{A_X(E)}{S_D(E)+ord_E(D)}$ w.r.t. X

i.e. in the expected vanishing order we consider sections on -mK that are "compatible" with D

Main advantage of considering $\tilde\delta_D$:

We do not change cohomology class, and this allows us to better study analytically and algebraically the properties of the function $Div_L(X)\ni D\longrightarrow\tilde\delta_D$.

Theorem
Let $L=-K$. Then
i) (X,L) is uniformly K-stable if and only if $\sup_{D\in Div_L(X)}\left(\tilde\delta_D^{1/m}\langle(L-D)^m\rangle\right)>(L^m)$;
ii) is "strongly" continuous. In particular the D-log Ding stability is an open condition;
iii) $\delta=\delta_0\leq(L^m)\inf_{D\in Div_L(X)}\frac{\tilde\delta_D}{\langle(L-D)^m\rangle}$.