On the existence of minimal models for generalized pairs

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$/\mathbb{C}$

varieties are normal projective

(1) The Minimal Model Program (MMP)

$$
\begin{tikzcd}
X \text{ : smooth} \arrow[r, dashed] & X' \text{ : mildly singular} \arrow[d, "f"] \\
 & Y
\end{tikzcd}
$$

Y

- If X: uniruled (i.e., covered by rational curves), then $f: X' \to Y$: Mori fibre space ($\dim Y < \dim X$, $-K_F$: ample)

$$\kappa(X) = -\infty$$

- If X: not uniruled, then X' is called a minimal model of X ($K_{X'}$: nef, i.e., $K_{X'} \cdot C \geq 0$, $\forall C \subseteq X$: curve) and the existence of f is predicted by the abundance

Conjecture:

$$\kappa(X) \geq 0$$

- If $\kappa(X) = \dim X$, then $K_{X'} \sim_{\mathbb{Q}} f^*(K_Y)$ and K_Y : ample

- If $0 \leq \kappa(X) < \dim X$, then f is a non-birational fibration and

$$K_{X'} \sim_{\mathbb{Q}} f^* A \ , \ A \subseteq Y : \text{ample}$$

F : general fibre $\Rightarrow \kappa(F) = 0$.

Upshot: building blocks

K : ample (canonically polarized)

K : trivial ((Calabi-Yau)

$-K$: ample (Fano)

Central problems:

$\exists$ minimal models (MM)

$\exists$ Mori fibre spaces (MFS)

[illegible]

Pairs: A pair (X,B) consists of a normal proj. var. X and an effective divisor B on X s.t. $K_X + B$ is $\mathbb{Q}$-Cartier

$\rightsquigarrow$ mild singularities, e.g.

- Kawamata log terminal (klt)
- log canonical (lc)

In this setting, the MMP has been confirmed in dim ≤ 3, but it is widely open in higher dimensions. However,

- $\exists$ MM for klt pairs of general type by [BCHM]
- $\exists$ MFS for lc pairs by [BCHM] and [Hashizume-Hu]

Generalized pairs: Roughly speaking, they are couples of the form $(X, B+M)$, where (X,B) is a usual pair and M has certain positivity properties (e.g., M: nef)

<u>DEF.</u>: A generalized pair (g-pair) consists of:

- X: normal proj. var.

- B : effective $\mathbb{Q}$-divisor on X
- $f: \underline{\underline{X'}} \to X$: proj. birational from a normal var. X', and
- $\underline{\underline{M'}} \subseteq X'$: nef,

such that $K_X + B + M$ is $\mathbb{Q}$-Cartier, where $M := f_* M'$.

- Notation : $(X, B+M)$.

- Motivation : Canonical Bundle Formula

Q: Are singularities preserved in an MMP?

$$\underset{\text{klt/lc}}{(X,B)} \dashrightarrow \underset{\text{klt/lc}}{(X',B')} : \mathrm{MM} \xrightarrow{f} \underset{?}{Y}$$

Consider:

(X,B) : klt/lc pair

klt/lc trivial fibration

$f : (X,B) \to Y$: fibration s.t.

$$K_X + B \sim_{\mathbb{Q}} f^*D,$$

where $D \subseteq Y$: $\mathbb{Q}$-Cartier.

$\Rightarrow \exists B_Y, M_Y \subseteq Y$ $\mathbb{Q}$-divisors s.t.

$$D \sim_{\mathbb{Q}} K_Y + B_Y + M_Y$$

and

$$K_X + B \sim_{\mathbb{Q}} f^*(\underbrace{K_Y + B_Y + M_Y}_{\text{generalized pair}}).$$

(2) Minimal Models and weak Zariski Decompositions:

If $\varphi : (X, B+M) \dashrightarrow (X', B'+M')$ is a MM of $(X, B+M)$ (i.e., φ: birational contraction and $K_{X'} + B' + M'$: nef) and if

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ {} & & {} \end{tikzcd}$$

$$X \overset{\varphi}{\dashrightarrow} X'$$

is a resolution of indeterminacies of φ, then

$$p^*(K_X+B+M) \sim_{\mathbb{Q}} \underbrace{q^*(K_{X'}+B'+M')}_{\text{nef}} + \underbrace{E}_{\geq 0}$$

· $\underline{\text{DEF}}$.: We say that a g-pair $(X, B+M)$ admits a weak Zariski decomposition

(WZD) if there is a projective bir. morphism $g: W \to X$ and a numerical equivalence

$$g^*(K_X+B+M) \equiv P+N$$

where P is nef and $N \geq 0$ $\mathbb{Q}$-Cartier

· THM 1 (Lazić, T.): Assume $\exists$ WZD for lc g-pairs of dim $\leq n-1$.

Let (X,B) be an lc pair of dim n s.t. K_X+B is pseudoeffective but $K_X+(1-\varepsilon)B$ is not pseudoeffective for any $\varepsilon>0$. Then (X,B) admits WZD.

PROOF (Sketch)

By running certain (carefully chosen) MMPs, we may reduce to the following situation:

$\exists g : X \to T$: fibration s.t.

- $0 < \dim T < \dim X$
- $K_X + B \sim_{\mathbb{Q}} g^* D_T$, where $D_T \subseteq T$ is $\mathbb{Q}$-Cartier.

$\xRightarrow{\text{CBF}}$ $\exists B_T, M_T \subseteq T$ s.t.

$$K_X + B \sim_{\mathbb{Q}} g^*(K_T + B_T + M_T) \quad (*),$$

where $(T, B_T + M_T)$ is an lc g-pair

(of $\dim T < \dim X$).

$\Longrightarrow (T, B_T + M_T)$ admits WZD

$\overset{(\star)}{\Longrightarrow} (X, B)$ admits WZD. $\blacksquare$

!, <u>REM</u>: An analogous statement holds for g-pairs $(X, B+M)$ s.t. $K_X + B + M$ is pseudoeffective but $K_X + B + (1-\varepsilon)M$ is not pseudoeffective for any $\varepsilon > 0$.

· <u>THM 2</u> (Lazić, T.): The existence of WZD for smooth var. of dim n implies the existence of WZD for lc g-pairs of dim n.

<u>Proof</u> (Idea): Use induction on the dimension and apply THM 1 and its variant to conclude. $\blacksquare$

THM 3 (Lazić, T.) ...

THM 3 (Lazić, T.): Assume the existence of minimal models for smooth var. of dim $n-1$.

Let $(X, B+M)$ be a $\mathbb{Q}$-fact. lc g-pair of dim n. Assume that either

(a) $(X, B+M)$ admits WZD, or

(b) K_X+B+M is not pseudoeffective.

Then there exists a (K_X+B+M)-MMP

which terminates with

- a minimal model of $(X,B+M)$, or
- a Mori fiber space of $(X,B+M)$.

In particular,

- if $n \leq 4$ and K_X+B+M is pseudo-effective, then $(X,B+M)$ has a minimal model.

- if $n \leq 5$ and K_X+B+M is not

pseudoeffective, then $(X, B+M)$ has a Mori fiber space.

Thm 4 (Lazić, T.): Assume the existence of minimal models for smooth varieties of dim $n-1$.

Let (X,B) be an lc pair of dim n such that X is uniruled and K_X+B is pseudoeffective. Then (X,B) has a

is pseudoeffective. Then (X,B) has a
minimal model.

In particular, pairs as above of $\dim \leq 5$ have minimal models.

! <u>REM</u>: An analogous statement holds for $\mathbb{Q}$-factorial lc g-pairs.

· <u>THM 5</u> (Lazić, T.): The existence of minimal models for smooth var. of dim n implies:

(i) the existence of minimal models for lc pairs of dim n, and

(ii) the existence of minimal models for $\mathbb{Q}$-fact. lc g-pairs of dim n.

<u>PROOF</u> (Sketch)

(i) similar to (ii)

(ii) $\exists$ MM for smooth var. of dim n

$\Longrightarrow$ $\exists$ MM for smooth var. of dim $\leq n$

$\Longrightarrow$ $\exists$ WZD for smooth var. of dim $\leq n$

$\overset{\text{THM 2}}{\Longrightarrow}$ $\exists$ WZD for lc g-pairs of dim $\leq n$

$\overset{\text{THM 3}}{\Longrightarrow}$ $\exists$ MM for lc g-pairs of dim n. $\blacksquare$