Parabolic Higgs bundles on toric varieties

Question: Let X be a compact Riemann surface.

Classify all holomorphic vector bundles $E \longrightarrow X$ of rank r on X.

Thm (Birkhoff, Grothendieck)

If $g(X)=0$ (i.e. $X \cong \mathbb{P}^1$), then $\mathcal{E} \cong \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_r)$

Thm (Atiyah)

If $g(X)=1$ (i.e. $X \cong E$, an elliptic curve), then

$$\left\{ \begin{array}{l} \text{indecomposable vector} \\ \text{bundles } E \to X \text{ of} \\ \text{degree } d \end{array} \right\} \xrightarrow{\sim} E$$

dep. on choice of base point

Thm (Narasimhan - Seshadri)

For $g(X) \geq 2$ there is a natural 1-1-correspondence

$$\left\{ \begin{array}{l} \text{stable vector bundles} \\ E \to X \text{ of rank } r \\ \text{\& degree } 0 \end{array} \right\} \xrightarrow{\sim} \mathrm{Rep}_{irr}(\pi_1(X), U(r))$$

Thm (Corlette, Simpson)

Let X be a smooth projective variety. There is a natural one-to-one correspondence

$$\left\{ \begin{array}{l} \text{stable Higgs bundles} \\ (E \to X,\ \phi : E \to E \otimes \Omega_X) \\ \text{of rank } r \text{ with } c(E) = 0 \end{array} \right\} \xrightarrow{\sim} \mathrm{Rep}_{irr}(\pi_1(X), GL(r))$$

This is very much a real-analytic correspondence, i.e. we have a real analytic isomorphism

$$\begin{array}{ccc} M^{s}_{Dol}(X) & \xrightarrow{\sim} & M^{*}_{Betti}(X) \\ \| & & \| \\ \{\text{stable Higgs bundles}\} & & \mathrm{Rep}_{irr}(\pi_1(X), GL(r)) /\!\!/ GL(r) \end{array}$$

If X = compact Riemann surface & $r = 1$ (so $GL(1) = \mathbb{G}_m$)

$$\mathrm{Jac}(X) \times H^0(X, \Omega_X) \xrightarrow{\sim} (\mathbb{C}^*)^{2g}$$

My question: What is the combinatorial content of the non-abelian Hodge correspondence?

Thm A Let X be a smooth complete toric variety with big torus T. Then there is a natural 1-1 correspondence

$$\left\{ \begin{array}{l} \text{stable parabolic Higgs bundles} \\ \left(E \to X, (E^{\rho}_{\alpha}), E \xrightarrow{\Phi} E \otimes \Omega_X^{\log}\right) \\ \text{of rank } r \text{ with } c(E_\alpha) = 0 \end{array} \right\} \xrightarrow{\sim} \mathrm{Rep}_{irr}\left(\pi_1^{\log}(X), GL(r)\right)$$

parabolic total Chern class

$\pi_1(T) = \mathrm{Hom}(\mathbb{G}_m, T) = N$

$$\Omega_X^{\log} = \Omega_X(\log D) \cong \mathcal{O}_X \otimes M \quad \text{where } M = \mathrm{Hom}(T, \mathbb{G}_m) = \mathrm{Hom}(N, \mathbb{Z})$$

Remarks:

- Thm A also works for complete toric orbifolds with some modifications (via canonical stacky resolution)
- The case of non-simplizial toric varieties is open.
- This might be a special case of a result of T. Mochizuki

Cor.: Let X be a smooth complete toric variety. Then there is a natural real-analytic isomorphism

$$\begin{tikzcd} M^s_{Dol}(X) \arrow[r, "\sim"] \arrow[d, phantom, "\cong"] & M^*_{Bet}(T) \arrow[d, phantom, "\|"] \\ & \mathrm{Hom}(N, GL(r)) /\!\!/ GL(r) \arrow[d, phantom, "\|\wr"] \\ M_{\mathbb{R}} \otimes (S^1)^r \arrow[r, "\sim"] & M \otimes (\mathbb{C}^*)^r \end{tikzcd}$$

$\mathbb{C}^* = \mathbb{R}_{\geq 0} \times S^1$

Def.: Let X be a smooth variety & $D = \sum_{i=1}^{k} D_i$ be an SNC-divisor on X. A parabolic bundle on (X, D) is a vector bundle E on X together with a collection of filtrations

$$\{E^i_\alpha\}_{\alpha \in [0,1]} \quad \text{of} \quad E|_D$$

with

(i) $E^i_0 = 0$

(ii) $E^i_1 = E|_D$

Let $X = X(\Delta)$ be a smooth toric variety with boundary divisor $D = \sum_{\rho\in\Delta(1)} D_\rho$. Let E be a vector bundle on X. Fix $V := E_1$ for $1 \in T$.

Then

$$\left\{\begin{array}{l}\text{parabolic structures}\\ \text{on } E\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Families of filtrations}\\ \{V_\alpha^\rho\}_{\alpha\in[0,1]} \text{ of } V\\ \text{satisfying (i) \& (ii)}\end{array}\right\}$$

Use evaluation map

$$E \longrightarrow E|_{D_\rho}$$

<u>Def.</u>: Let V be a fin. dim. $\mathbb{C}$-vector space.

A <u>non-Archimedean norm</u> on V is a map

$$\|.\|: V \longrightarrow \mathbb{R}_{\geq 0}$$

s.t.

(i) $\|v\| = 0 \iff v = 0$

(ii) $\|\lambda v\| = \|v\| \quad \forall v \in V, \lambda \in \mathbb{C}$

(iii) $\|v+w\| \leq \max\{\|v\|, \|w\|\} \quad \forall v, w \in W$

A non-Archimedean norm $\|.\|$ on V is <u>bounded</u>, if $\|v\| \leq 1 \ \forall v \in V$.

Note: There is a natural 1-1 equivalence

$$\left\{\begin{matrix}\text{bounded}\\ \text{non-Arch}\\ \text{norms on } V\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{filtrations}\\ \{V_\alpha\}_{\alpha\in[0,1)} \text{ of } V\\ \text{with (i) \& (ii)}\end{matrix}\right\}$$

$$\|.\| \longmapsto V_\alpha := \{v \in V \mid \|v\| \leq \alpha\}$$

Thm B Let X be a smooth toric variety with big torus T. There is a natural equivalence

$$\left\{\begin{matrix}\text{parabolic bundles}\\ (E\to X, (E^\rho_\alpha)) \text{ on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{vector bundles } E\to X\\ \text{with } T\text{-invariant}\\ \text{non-Arch. norms}\end{matrix}\right\}$$

smoothness of X is not needed !

continuous on $E^{an} \to X^{an}$ in the sense of Berkovich

see e.g. Chambert-Loir etc.

Example: (Klyachko) Let $E \to X$ be a toric vector bundle. Then for $\rho \in \Delta(1)$ and $u \in N = \mathrm{Hom}(\mathbb{G}_m, T)$ the minimal generator of ρ

$$\|v\|_\rho := e^{-\operatorname{ord}_{D_\rho}\left(\lim_{t\to 0} \rho(t)\cdot v\right)} \qquad v \in V$$

defines non-Arch. norms $\|.\|_\rho$ on V (and thus on E).

Remarks: • Thm B generalizes Klyachko's classification of toric vector bundles

• Let $\mathcal{N}(V)$ be the space of non-Archimedean norms on V. Then a parabolic structure is the same thing as a piecewise $\mathbb{R}$-linear map

$$\Delta \longrightarrow \mathcal{N}(V)$$

up to choice of a frame.

$\mathcal{N}(V) \cong$ Bruhat-Tits building of $GL_n(\mathbb{C})$

$|| \; GL_n^{trop}(\mathbb{C})$

see the work of Kaveh-Manon

↳ Obtain piecewise linear maps

$$\psi_i : |\Delta| \longrightarrow \mathbb{R} \qquad (i=1,\dots,n)$$

Def.: Let X be a smooth variety & $D=\sum_{i=1}^{k} D_i$ an SNC-divisor. A parabolic Higgs bundle on X is a parabolic vector bundle $(E,(E^i_\alpha))$ together with

$$\theta : E \longrightarrow E \otimes \Omega^1_X(\log D)$$

that is compatible with the filtrations & fulfills

$$\theta \wedge \theta = 0$$

Recall: Let $X = X(\Delta)$ be a smooth & complete toric variety. Then there is a natural Isomorphism

$$H^*(X, \mathbb{R}) \xrightarrow{\sim} PP^*(\Delta)_{\mathbb{R}}$$

$\shortparallel$

piecewise polynomial functions on Δ

$\shortparallel$

$$\left\{ f : |\Delta| \to \mathbb{R} \;\middle|\; f|_\sigma \in \operatorname{Sym} M_{\sigma,\mathbb{R}} \quad \forall \sigma \in \Delta \right\}$$

where $M_\sigma = M / M \cap \sigma^\perp$

Def.: [Payne (for toric vector bundles), U]

Let e_i be the i^{th} elementary symmetric polynomial in r variables. Define the parabolic total Chern class $c(E, (E^\rho_\alpha)) \in H^*(X, \mathbb{R})$ of a parabolic bundle $(E, (E^\rho_\alpha))$ by

$$c(E, (E^\rho_\alpha)) := 1 + \underbrace{e_1(\psi_1, \dots, \psi_r)}_{c_1} + \dots + \underbrace{e_r(\psi_1, \dots, \psi_r)}_{c_r}$$

$c_1 = $ parabolic degree

In order to prove Thm A we need the following Linear Algebra Lemma

<u>Lemma:</u> Let $k_1+\dots+k_l = r$ an ordered partition & write $\mathcal{F}L(k_1,\dots,k_l) = \{0 \subseteq V_1 \subseteq \dots \subseteq V_l = \mathbb{C}^r \mid \dim V_i/V_{i-1} = k_i\}$ for the flag variety of signature $(k_1,\dots,k_n)$.

Then there is a natural isomorphism

$$\mathcal{F}l((k_1,\dots,k_l) = U(r)/_{U(k_1)\times\dots\times U(k_l)}$$

<u>Sketch of proof for Thm A:</u>

Consider $U_\rho \subseteq X$ for $\rho \in \Delta(1)$. Then $U_\rho \cong (\mathbb{C}^*)^{n-1} \times \mathbb{A}^1$. Let γ be a simple loop around the boundary. Given $\alpha: \pi_1(T) \longrightarrow U(r)$, then

$$\alpha([\gamma]) = \left[\begin{array}{c|c} \begin{matrix} e^{2\pi i \alpha_1} & & \\ & \ddots & \\ & & e^{2\pi i \alpha_1} \end{matrix} \;\; k_1 & 0 \\ \hline 0 & \ddots \\ & k_l \;\; \begin{matrix} e^{2\pi i \alpha_l} & & \\ & \ddots & \\ & & e^{2\pi i \alpha_l} \end{matrix} \end{array}\right]$$

Lemma

$\Rightarrow$ filtration $\quad 0 \subseteq \underset{\alpha_1}{\overset{\uparrow}{V_1}} \subseteq \cdots \subseteq \underset{\alpha_l}{\overset{\uparrow}{V_l}} = \mathbb{C}^r$

$c(E, E^{\varrho}_{\alpha}) = 0$

$\Rightarrow$ parabolic structure

Higgs field: QR-decomposition

stability: usual yoga with parabolic degrees ▨

For me the case of a toric variety is really a test case for a yet-to-be-developed logarithmic non-abelian Hodge-correspondence along the lines of the following:

<u>Vague conj.</u>:

Let X be a logarithmic curve. There is a natural equivalence

$$\left\{\begin{array}{l}\text{stable parabolic Higgs} \\ \text{bundles on } X \text{ of rank } r \\ \text{with } c_{Par}(-) = 0\end{array}\right\} \xrightarrow{\sim} \mathrm{Rep}_{irr}\left(\pi_1^{log}(X), GL(r)\right)$$

that is induced by the usual parabolic Simpson correspondence on each component.