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MORI DREAM PAIRS & $\mathbb{C}^*$-ACTIONS

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IDEA: Understand the relation between particular $\mathbb{C}^*$-actions & 2-dim Mori Dream Regions

We work / $\mathbb{C}$ and consider Y a normal projective variety.

A small modification of Y is a birational map

$$\phi : Y \dashrightarrow Y' \quad (Y' \text{ normal projective})$$

which is an iso. in codimension 1.

(We say that ϕ is SQM small $\mathbb{Q}$-factorial modification adding the requirement that both Y, Y' are $\mathbb{Q}$-factorial)

Let us fix a finite set of effective Cartier divisors on Y $L_1, \dots L_k$

$$R(Y; L_1, \dots, L_k) := \bigoplus_{m_1, \dots m_k \in \mathbb{N}} H^0(Y, \mathcal{O}_Y(m_1 L_1 + \dots + m_k L_k))$$

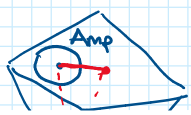
this is a multigraded $\mathbb{C}$-algebra

DEF ① Let $C = \langle L_1, \dots L_k \rangle$ rational polyhedral cone in $\mathrm{CDiv}(Y)_{\mathbb{Q}}$, $L_i \geq 0$. C is a MORI DREAM REGION if $R(Y; L_1, \dots L_k)$ is a f.g. $\mathbb{C}$-algebra [OKAWA]

② (Special case) Let Y normal projective, $L_1, L_2 \in \mathrm{CDiv}(Y)$ s.t.
- L_1 ample
- $R(Y, L_2)$ finitely generated &

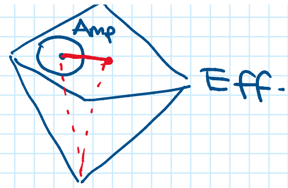
$$\phi : Y \dashrightarrow Y' := \mathrm{Proj}(R(Y; L_2)) \text{ is a small modification.}$$

THEN: we say that (L_1, L_2) is a MORI DREA PAIR if $\langle L_1, L_2 \rangle$ is a MDR



Eff.

$\langle L_1, L_2 \rangle$ is a MDR



Let's see how this connects to $\mathbb{C}^*$-actions.

STEP (1) Białynicki - Birula DECOMPOSITION.

Fix (X, L) a polarized pair, w/ X normal projective, L ample.
Assume that the pair admits a $\mathbb{C}^*$-action.
then the action admits a linearization on L

$$\begin{tikzcd} \mathbb{C}^* \times L \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[d] \\ \mathbb{C}^* \times X \arrow[r] & X \end{tikzcd}$$

Given the action, consider the set

$\mathcal{Y}$ of irred. comp. components of the fixed point locus.

& assign a weight to each $Y \in \mathcal{Y}$ i.e.

$$\mu_L(Y) \in M(\mathbb{C}^*) := \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \simeq \mathbb{Z}$$

call such values critical values.

B-B: is a finite chain of values

$a_0 < a_1 < \dots < a_r$ and we can define

$a_r - a_0 =: \delta$ bandwidth of the action. r is the criticality.

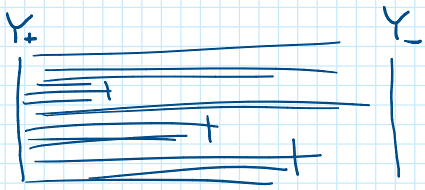
$$Y_i := \bigsqcup_{\substack{Y \in \mathcal{Y} \\ \mu_L(Y) = a_i}} Y$$

it can be proved that Y_0 & Y_r are irreducible
$Y_0 = Y_+$, $Y_r = Y_-$

For all $Y \in \mathcal{Y}$ we denote

$$X^+(Y) := \{ x \in X \mid \lim_{t \to 0} tx \in Y \}$$
$$X^-(Y) := \{ x \in X \mid \lim_{t \to \infty} tx \in Y \}$$

B-B cells.



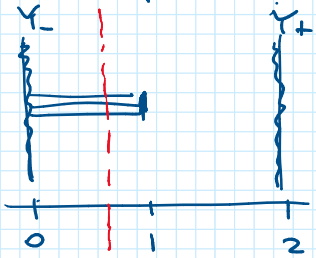
Known: $X^+(Y_+)$ & $X^-(Y_-)$ are dense open subsets on X

How to construct GIT quotients?

For any index $i = 0, \dots, r-1$

$\mathcal{Y} = \{ Y \in \mathcal{Y} \mid \mu_L(Y) \le a_i \} \sqcup \{ Y \in \mathcal{Y} \mid \mu_L(Y) \ge a_{i+1} \}$

$X^s(i, i+1) := X \setminus \left(\bigsqcup_{Y \in \mathcal{Y}_-} X^+(Y) \sqcup \bigsqcup_{Y \in \mathcal{Y}_+} X^-(Y) \right)$ is open in X



the GIT quotients are defined as

$\mathcal{G}X(i, i+1) := X^s(i, i+1) /\!\!/ \mathbb{C}^*$

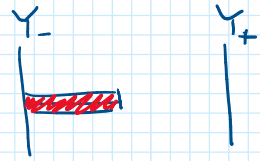
$\parallel$ FIX $\tau \in (a_i, a_{i+1}) \in \mathbb{Q}$

$\operatorname{Proj} \bigoplus_{\substack{m \ge 0 \\ m\tau \in \mathbb{Z}}} H^0(X, mL)_{m\tau}$ $\rightsquigarrow$ weight of the $\mathbb{C}^*$-action.

<u>Def</u>: (X, L) polarized w/ $\mathbb{C}^*$ action of criticality r

* the action is of B-type if
  $\mathcal{G}X(0,1) \longrightarrow Y_-$ $\qquad$ $\mathcal{G}X(r-1, r) \longrightarrow Y_+$ are isom.

* the action is a BORDISM if it is of B-type
  and $\overline{X^{\pm}(Y)}$ does not contain codim. 1- subvarieties
  for every Y inner component



<u>RMK</u>: Given a bordism, the natural map
$\psi : \mathcal{G}X(0,1) \longrightarrow \mathcal{G}X(r-1, r)$ is a small modification.

<u>RECALL</u> (WORS) Given a birational map $\phi : Y_- \dashrightarrow Y_+$

A GEOMETRIC REALIZATION of ϕ is a normal projective variety X w/ a $\mathbb{C}^*$-action of B-type w/ sink & source Y_- & Y_+ & ϕ is the natural induced map.

<u>THE CORRESPONDENCE</u>:

(A) MDP $\Rightarrow$ $\mathbb{C}^*$-actions

Let $\phi : Z_1 \dashrightarrow Z_2$ a small modification.
L_1 ample on Z_1, $L_2 = \phi^*(\bar{L})$ $\bar{L}$ ample on Z_2 s.t.
$A := \bigoplus_{a, b \ge 0} H^0(Z_1, aL_1 + bL_2)$ is finitely generated

<u>THM</u> (BRUS) $\exists$ a normal PROJECTIVE VARIETY X w/ a $\mathbb{C}^*$-action

st
(i) the action is a bordism
(ii) the sink is Z_1 & the source is Z_2
(iii) Φ coincides w/ ψ

PF construction of X

Let $H := \mathrm{Hom}(\mathbb{Z}(L_1, L_2), \mathbb{C}^*)$ is a complex 2-dim torus acting naturally on A

$M(H) = \mathbb{Z}(L_1, L_2)$ the character lattice

$\alpha \in M(H)^\vee$ a 1-parameter subgroup s.t. $\alpha_{1,2} := \alpha(L_i) > 0$ (and coprime)

the choice of α induces a natural subtorus $H' \subseteq H$ 1-dim acting on A

$$\Rightarrow \quad A = A^\alpha := \bigoplus_{m \geq 0} A^\alpha_m \quad \text{w/} \quad A^\alpha_m := \bigoplus_{\substack{m_i \in \mathbb{Z}_{\geq 0} \\ \alpha(m_1 L_1 + m_2 L_2) = m}} H^0(Y_1, m_1 L_1 + m_2 L_2)$$

$$X^\alpha := \mathrm{Proj}\, A^\alpha$$

$$H' \xrightarrow{\alpha} H \longrightarrow H'' \qquad H'' = H/H' \circlearrowright X^\alpha$$

BORDISM $\Rightarrow$ MDP

Let (X, L) polarized pair w/ a $\mathbb{C}^*$ action. X normal, $\mathbb{Q}$-fact proj, L ample cartier.

THM the induced birational map $\psi : Y_- \dashrightarrow Y_+$ is a small $\mathbb{Q}$-factorial modification & the pair (L_-, L_+) w $L_- = L|_{Y_-}$ & $L_+ = \psi_*^{-1} L|_{Y_+}$ is a MDP whose natural associated map is ψ.