Varieties of general type with small volume

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Volume

- (Hacon-McKernan, Takayama, and Tsuji) For each integer $n > 0$, $\exists$ a constant r_n s.t. for any smooth complex projective variety X of general type with dimension n, the map $\varphi_{|mK_X|} : X \dashrightarrow P^{h^0(mK_X)-1}$ is a birational embedding for $m \geq r_n$.
- Volume of X: $vol(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. $vol(X) = K_X^n$ if K_X is ample. (Also when X is a normal projective variety with at worst canonical singularities and with nef K_X.)
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Smooth varieties of general type in low dimensions

- $dim = 1$, $r_1 = 3$, $a_1 = 2$.
- $dim = 2$, $r_2 = 5$ (by Bombieri), $a_2 = 1$. The extreme case: a general hypersurface $X_{10} \subset P(1,1,2,5)$.
- $dim = 3$, $r_3 \leq 57$, $a_3 \geq 1/1680$ (by J. Chen and M. Chen). The smallest known volume is 1/420 (Iano-Fletcher): a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$. $|mK_X|$ is birational $\Leftrightarrow m = 23$ or $m \geq 27$.
- $dim = 4$, the smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$ (by Brown and Kasprzyk).

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In high dimensions

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

1. $\exists$ *a smooth complex projective n-fold of general type with volume less than* $1/n^{(n \log n)/3}$.
2. $\exists$ *a smooth complex projective n-fold X of general type s.t. the linear system* $|mK_X|$ *does not give a birational embedding for any* $m \leq n^{(\log n)/3}$.

Ballico, Pignatelli, and Tasin found smooth n-folds of general type with volume about $1/n^n$, and s.t. $|mK_X|$ does not give a birational embedding for m at most a constant times n^2.

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Noether-type inequality

- Surfaces of general type: $vol(X) \geq 2p_g - 4$, where the geometric genus $p_g = h^0(X, K_X)$.
- (M. Chen and Z. Jiang) For every positive integer n, $\exists\ a_n > 0$, $b_n > 0$ s.t. $vol(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type.
- (J. Chen, M. Chen, and C. Jiang) 3-folds of general type: $vol(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$.(optimal constants)
- In high dimensions, our examples show that $a_n < 1/n^{(n \log n)/3}$ for all sufficiently large n.
  A simple approach to this implication is to take the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai

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well-formed

- The weighted projective space $Y = P(a_0, \ldots, a_n)$ is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$, has trivial stabilizer group in codimension 1.)
- A general hypersurface of degree d is well-formed $\Leftrightarrow$ $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n) | d$ for all $i < j$, and $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.
- Reflexive sheaf $O(m)$ is a line bundle $\Leftrightarrow$ m is a multiple of every weight a_i.
- The intersection number $\int_Y c_1(O(1))^n = 1/a_0 \cdots a_n$.

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Reid-Tai criterion for quotient singularities

For a positive integer r, let A^n/μ_r be the cyclic quotient singularity of type $\frac{1}{r}(a_1, \ldots, a_n)$ over a field, meaning that the group μ_r of rth roots of unity acts by $\zeta(x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n)$.

Assume that this description is well-formed in the sense that $\gcd(r, a_1, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for $j = 1, \ldots, n$. Then A^n/μ_r is canonical (resp. terminal) $\Leftrightarrow$

$$\sum_{j=1}^{n} (ia_j \bmod r) \geq r$$

(resp. $> r$) for $i = 1, \ldots, r-1$.

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Assume that this description is well-formed in the sense that $\gcd(r, a_1, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for $j = 1, \ldots, n$. Then A^n/μ_r is canonical (resp. terminal) $\Leftrightarrow$

$$\sum_{j=1}^{n} (ia_j \bmod r) \geq r$$

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It suffices for Y to be canonical or terminal at each coordinate point, $[0, \dots, 0, 1, 0, \dots, 0]$.

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A general hypersurface of degree d in $P(a_0, \ldots, a_n)$ is quasi-smooth $\Leftrightarrow$

- *either (1) $a_i = d$ for some i,*
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compute the volume

A general hypersurface X of degree $d = (l+3)k(k+1)$ in $Y = P(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)})$.

- Adjunction formula holds:

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a)\ \text{X is well-formed.} \\ (b)\ \text{X is quasi-smooth since} \\ \text{d is a multiple of all the weights.} \end{cases}$$

  Thus $K_X = O_X(1)$ ample. So $vol(X) = K_X^n$, which is d divided by the product of all weights of Y.
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Let b, l, k be integers with $b \geq 2$, $l \geq 0$, and $k \geq 2b - 2$. For each subset I of $\{0, \ldots, b-1\}$, define (with j running through $0, 1, \ldots, b-1$):

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Terminal Fano varieties.

- (Birkar) For each integer $n > 0$, $\exists$ a constant s_n s.t. for every terminal Fano n-fold X, $|-mK_X|$ gives a birational embedding for all $m \geq s_n$;
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- (J. Chen and M. Chen) The optimal cases:
  $dim = 2$, $X_6 \subset P(1,1,2,3)$ with volume 1,
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Adding two more weights equals to 1 in the weighted projective space Y.

Theorem (B. Totaro, C. Wang)

For every sufficiently large positive integer n,

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Fujita's conjecture: for every smooth complex projective variety X of dimension n with an ample line bundle A, $K_X + (n+2)A$ is very ample.

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In high dimensions:

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- Construct weighted projective space $P(a_0, \ldots, a_{n+1})$.
- Sylvester's sequence: $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807, \ldots$ and $c_{n+1} = c_n(c_n - 1) + 1$.
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- From a criterion for quasi-smoothness proved by Iano-Fletcher, we get a sufficient condition for quasi-smooth:

For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $P(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \geq a_i$ for every i and there is a positive integer r such that:

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Our construction of klt varieties with ample canonical class:

- Sylvester's sequence $\{c_i\}$.
- $n \geq 2$. Let $y = c_{n-1} - 1$ and
  $a_2 = y^3 + y + 1,$
  $a_1 = y(y+1)(1+a_2) - a_2,$
  $a_0 = y(1 + a_2 + a_1) - a_1.$
- Let $x = 1 + a_0 + a_1 + a_2,$
  $d = yx = c_0 \cdots c_{n-2} x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y,$
  and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2} x$ for $0 \leq i \leq n-2$.
- $X \subset P(a_0, \ldots, a_{n+1})$ is a general hypersurface of degree d.

sketch of proof when $r = 3$

- X is klt since it has only cyclic quotient singularities.
- X is quasi-smooth since $d - a_2 = (y^2 + 1)a_1, d - a_1 = (y + 1)a_0, d - a_0 = (y^4 + 3y - 1)a_2$. (by Lemma)

$$K_X = O_X(d - \sum a_i) \Leftarrow \begin{cases} (a) \text{ X is well-formed} \\ (b) \text{ X is quasi-smooth} \end{cases}$$

- $vol(K_X) = vol(O_X(1)) = \frac{d}{a_0 \cdots a_{n+1}} = \frac{1}{y^{n-3}x^{n-2}a_0 a_1 a_2}$

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  Thus $vol(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.
- There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$. This implies the crude statement that $vol(K_X) < \frac{1}{2^{2^n}}$ for all $n \geq 2$.

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Better klt varieties with ample canonical class

For any odd number $r \geq 3$ and any dimension $n \geq r - 1$, we give an example with weights chosen to satisfy a cycle of r congruences.

- $\frac{\log(vol(K_X))}{\log(vol(K_Y+\Delta))} \to \frac{2^r-1}{2^r}$ as $n \to \infty$.
- For $r = 3$, this is the example above.
- When $r = 5$, $n = 4$, it is a a general hypersurface of degree 147565206676 in $P(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361)$ with $\doteq 7.4 \times 10^{-45}$. (Better)

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Thank you!