Fano varieties with extreme behavior

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- A normal projective variety X is Fano if the anti-canonical divisor $-K_X$ is ample.
- The *Fano index* of X is:

$$\mathrm{FI}(X) := \max\{m \in \mathbb{Z}_{>0} | -K_X \sim_{\mathbb{Q}} mA, \text{ where } A \text{ is a Weil divisor}\}.$$

Kobayashi and Ochial: $\mathbb{P}^n$ has the biggest Fano index $n+1$ among all smooth Fano varieties. $-K_{\mathbb{P}^n} = (n+1)\mathcal{O}(1)$

- $\mathbb{Q}$-Fano variety:
Fano variety
only terminal $\mathbb{Q}-$factorial singularities;
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Prokhorov: for a $\mathbb{Q}-$Fano threefold X

- the Fano index belongs to $\{1, \ldots, 11, 13, 17, 19\}$.
- $\mathrm{FI}(X) = 19 \iff X \simeq \mathbb{P}^3(7,5,4,3)$;
  $-K_X = \mathcal{O}(7+5+4+3) = \mathcal{O}(19)$ and $\mathcal{O}(1)$ is Weil divisor.
- $\mathrm{FI}(X) = 17 \iff X \simeq \mathbb{P}^3(7,5,3,2)$
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Weighted Projective Spaces

- For $a_0, \ldots, a_N \in \mathbb{Z}_{>0}$,
  the WPS $X = \mathbb{P}^N(a_0, \ldots, a_N)$ is the quotient variety $(\mathbb{A}^{N+1} \setminus 0)/\mathbb{G}_m$, where the multiplicative group $\mathbb{G}_m$ acts by $t(x_0, \ldots, x_N) = (t^{a_0}x_0, \ldots, t^{a_N}x_N)$.
- WPS X is called *well-formed*
  $\iff$ analogous quotient stack $[(A^{n+1} - 0)/\mathbb{G}_m]$ has trivial stabilizer group in codimension 1.
  $\iff \gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$ for each i.

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X is well-formed WPS

- $\mathcal{O}(c)$: the sheaf associated to a Weil divisor on X for an integer c.
- $\mathcal{O}(c)$ is a line bundle $\Longleftrightarrow$ every weight a_i is a factor of c.
- the canonical divisor $K_X = \mathcal{O}(-a_0 - \cdots - a_N)$.
- a Weil divisor is ample if some positive multiple of it is an ample Cartier divisor.
- the ample Weil divisor $\mathcal{O}(1)$ has volume $\frac{1}{a_0 \cdots a_N}$.

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Reid-Tai criterion

Theorem

- *The group of* r*th roots of unity* μ_r *acts on affine space* $\mathbb{A}^s$ *by* $\zeta(t_1, \dots, t_s) = (\zeta^{b_1} t_1, \dots, \zeta^{b_s} t_s)$.
- *Quotient* $\mathbb{A}^s/\mu_r$ *is a cyclic quotient singularity of type* $\frac{1}{r}(b_1, \dots, b_s)$.
- *Assume that* $\gcd(r, b_1, \dots, \widehat{b_i}, \dots, b_s) = 1$ *for all* $i = 1, \dots, s$ *(this description is well-formed).*
  Then the quotient singularity is canonical (resp. terminal)

$$\iff \sum_{k=1}^{s} tb_k \bmod r \geq r$$

(resp. $> r$*) for all* $t = 1, \dots, r-1$.

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Sylvester's sequence

$s_0 = 2$, and $s_n = s_{n-1}(s_{n-1} - 1) + 1$ for $n \geq 1$.

First few terms: 2,3,7,43,1807.

- $s_n > 2^{2^{n-1}}$ for all n, grows doubly exponential with respect to n.
- $s_n = s_0 \cdots s_{n-1} + 1$, hence pairwise coprime.
- $\frac{1}{s_0} + \frac{1}{s_1} + \cdots + \frac{1}{s_{n-1}} = 1 - \frac{1}{s_n - 1}$.

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Canonical singularities

Lower dimensions
Del Pezzo surface $X = \mathbb{P}^2(3, 2, 1)$ has Fano index 6 which is the largest Fano index among all weighted projective planes with canonical singularities (Brown and Kasprzyk).

I show the result in greater generality:

Proposition (Wang2023)

Among all canonical del Pezzo surfaces, the WPS $X = \mathbb{P}^2(3, 2, 1)$ has the largest Fano index 6.

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- $n = 3$, $X = \mathbb{P}^3(33, 22, 6, 5)$ has Fano index 66.
  It is the largest Fano index among all WPS of dimension 3 with canonical singularities (Averkov, Kasprzyk, Lehmann, Nill).
- $n = 4$, $X = \mathbb{P}^4(1743, 1162, 498, 42, 41)$ has Fano index 3486.
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Theorem (Wang2023)

For each integer $n \geq 2$, let

- $h = (s_{n-1} - 1)(2s_{n-1} - 3)$,
- $a_i = h/s_{n-i}$ *for* $2 \leq i \leq n$,
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Conjecture: this is the example of the largest possible Fano index among all Fano $n-$folds with canonical singularities. True for $\dim = 2$

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$X = \mathbb{P}^n(a_n, \ldots, a_0) = \mathbb{P}^n(h/s_0, \ldots, h/s_{n-2}, s_{n-1} - 1, s_{n-1} - 2)$ *is well-formed with canonical singularities and with Fano index h.*

Conjecture: this is the example of the largest possible Fano index among all Fano $n-$folds with canonical singularities.

True for $\dim = 2$

generalize to higher dimensions

Theorem (Wang2023)

For each integer $n \geq 2$, let

- $h = (s_{n-1} - 1)(2s_{n-1} - 3)$,
- $a_i = h/s_{n-i}$ *for* $2 \leq i \leq n$,
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- $n = 2$, $X = \mathbb{P}^2(3, 2, 1)$, $\mathrm{FI}(X) = 6$,
- $n = 3$, $X = \mathbb{P}^3(33, 22, 6, 5)$, $\mathrm{FI}(X) = 66$,
- $n = 4$, $X = \mathbb{P}^4(1743, 1162, 498, 42, 41)$, $\mathrm{FI}(X) = 3486$.

Let $h_n = (s_n - 1)(2s_n - 3)$. We have $h = h_{n-1}$ above.

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Index of Calabi-Yau varieties

A normal projective variety X is **Calabi-Yau** if its canonical divisor $K_X \sim_{\mathbb{Q}} 0$.

The **index** of X is the smallest positive integer m with $mK_X \sim 0$.

- A smooth CY surface of index 6 : a "bielliptic" surface $(E_1 \times E_2)/\mu_6$, where E_i is a smooth elliptic curve.
- A smooth CY 3-fold of index 66 : $(Z \times E)/\mu_{66}$, where Z is a smooth $K3$ surface.

Calabi-Yau pair (X, D):
a normal projective variety X,
an *effective* $\mathbb{Q}$-divisor D on X such that $K_X + D \sim_{\mathbb{Q}} 0$.

Klt Calabi-Yau pairs with standard coefficients

(X, D): a klt Calabi-Yau pair with standard coefficients ($1 - \frac{1}{b}$, $b \in \mathbb{Z}_{>0}$), and index m.
The (global) **index-1 cover** of (X, D) is a projective variety X' with canonical Gorenstein singularities s.t. $K_{X'} \sim 0$.

Here (X, D) is the quotient of X' by an action of the cyclic group μ_m such that μ_m acts faithfully on $H^0(Y, K_{X'}) \cong \mathbb{C}$. (In dim 2, purely non-symplectic action)

$\pi : X' \to X$, $K_{X'} = \pi^*(K_X + D)$.

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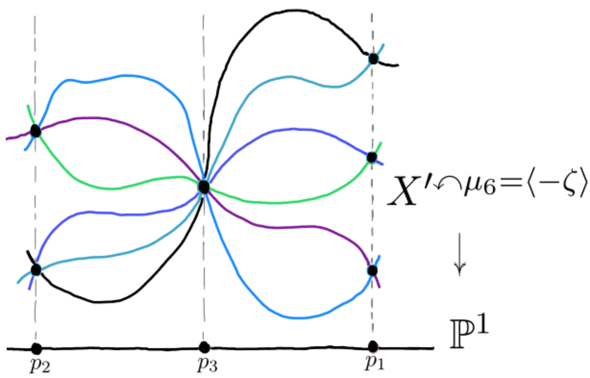
Klt CY pair in dim. 1 of the largest index

The unique klt CY pair of index 6: $(\mathbb{P}^1, \frac{1}{2}p_1 + \frac{2}{3}p_2 + \frac{5}{6}p_3)$.
Index cover X' is the *unique* elliptic curve $\mathbb{C}/\mathbb{Z}[\zeta]$ over $\mathbb{C}$, where ζ is a cubic root of unity. $K_{X'} = \pi^*(K_{\mathbb{P}^1} + \frac{1}{2}p_1 + \frac{2}{3}p_2 + \frac{5}{6}p_3)$.



Calabi-Yau variety with small volume

For $n \in \mathbb{Z}_{\geq 0}$, let $h_n = (s_n - 1)(2s_n - 3)$ and $d = 2s_n - 2$, the hypersurface $\widehat{X'_{h_n}} \subset \mathbb{P}(h_n/s_0, \ldots, h_n/s_{n-1}, s_n - 1, s_n - 2)$ defined by $x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_n - 1} + x_n^{d-1} + x_n x_{n+1}^d = 0$ has $\mathrm{vol}(\mathcal{O}_{\widehat{X'_{h_n}}}(1)) < 1/2^{2^n}$.

It is the conjecturally **minimum volume** among all canonical Calabi-Yau n-folds with an ample Weil divisor $\mathcal{O}(1)$. (ETW 2021)

Mirror Symmetry

A **potential** $W = \sum_{i=1}^{n} \prod_{i=1}^{n} x_i^{a_{ij}}$ is a sum of n monomials in n variables which is described by a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$. The potential is called **invertible** if A is invertible which determines a hypersurface in a WPS.

- charge q_i: the sum of the entries of the $i-$th row of A^{-1},
- d: the least common denominator of q_i and $w_i := dq_i$,
- $W = 0$ defines a degree d hypersurface in $\mathbb{P}(w_1, \ldots, w_n)$.

Let $\widehat{W}$ be the potential described by the transpose matrix of A.

The hypersurfaces defined by $W = 0$ and $\widehat{W} = 0$ are Berglund-Hübsch-Krawitz (BHK) mirror to each other.

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(BHK) mirror

$$\widehat{W} : x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_n-1} + x_n^{d-1} + x_n x_{n+1}^d$$

$$\begin{pmatrix} 2 & & & & \\ & 3 & & & \\ & & \ddots & & \\ & & & d-1 & \\ & & & 1 & d \end{pmatrix} \xrightarrow{\text{transpose}} \begin{pmatrix} 2 & & & & \\ & 3 & & & \\ & & \ddots & & \\ & & & d-1 & 1 \\ & & & & d \end{pmatrix}$$

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For $n \in \mathbb{Z}_{\geq 0}$, $h_n = (s_n - 1)(2s_n - 3)$, $d = 2s_n - 2 = 2s_0 \cdots s_{n-1}$

- The hypersurface $X'_d \subset \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ defined by $x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1} x_{n+1} + x_{n+1}^d = 0$ is quasi-smooth of dimension n, canonical, and has $K_{X'} \sim 0$;
- $\widehat{X'_{h_n}}$ is the Berglund-Hübsch-Krawitz (BHK) mirror of X'.

$$X'_d \xleftrightarrow{\text{mirror}} \widehat{X'_{h_n}}$$

There is a easy combinatorial way to compute big cyclic group action on the hypersurface defined by a potential.

- μ_{h_n} acts $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ by $\zeta[x_0 : \cdots : x_{n+1}] = [\zeta^{d/(2s_0)} x_0 : \zeta^{d/(2s_1)} x_1 : \cdots : \zeta^{d/(2s_{n-1})} x_{n-1} : x_n : \zeta^{d/2} x_{n+1}]$.
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Calabi-Yau pairs of large index (simplified description)

Theorem (ETW 2022)

For an integer $n \geq 2$, let

- $X = \mathbb{P}^n(d^{(n-1)}, d-1, 1)$ *with* $d = 2s_n - 2$ *and coordinates* $y_1, \ldots, y_{n+1}$;
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Then (X, D) *is a klt Calabi-Yau pair of dimension* n *with standard coefficients of index* $h_n = (s_n - 1)(2s_n - 3) > 2^{2^n}$.

Conjecture: this is the example of largest index.
True for dim$= 2$.

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Dimension 2

$$(X, D) = (\mathbb{P}^2(12, 11, 1), \tfrac{1}{2}D_0 + \tfrac{2}{3}D_1 + \tfrac{10}{11}D_2).$$

Index-1 cover:

$X'_{12} \subset \mathbb{P}(6, 4, 1, 1)$ given by $x_0^2 + x_1^3 + x_2^{11}x_3 + x_3^{12} = 0$ acted by μ_{66}.

$\widehat{X'_{66}} \subset \mathbb{P}(33, 22, 6, 5)$ given by $x_0^2 + x_1^3 + x_2^{11} + x_2x_3^{12} = 0$

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Proposition (Wang2023)

Among all canonical del Pezzo surfaces, the WPS $X = \mathbb{P}^2(3, 2, 1)$ has the largest Fano index 6.

Fano index of $\mathbb{P}^2(3, 2, 1)$ is $3 + 2 + 1 = 6$.

Lemma (1)

Let X be a smooth projective surface and Y be the blow-up of X at a point. Then K_Y is always primitive, i.e., then there exists no element $A \in \mathrm{Pic}(Y)$ such that $K_Y \sim_{\mathbb{Q}} mA$ for some integer $m \geq 2$.

Proof: We have $K_Y \cdot E = -1$, where E is the exceptional divisor of the blow up.

$K_Y \sim_{\mathbb{Q}} mA$ for some positive integer m and $A \in \mathrm{Pic}(Y)$

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$m(A \cdot E) = -1$.

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Lemma (2)

For a canonical del Pezzo surface S with Picard number one, the Fano index $\mathrm{FI}(S) \leq 6$.

Idea: Use classification of canonical (equivalent to Gorenstein in dimension 2) del Pezzo surfaces S with Picard number one, and canonical volume $(-K_S)^2$. (Miyanishi, Zhang)

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Sketchs:

- Assume that $-K_S \sim_{\mathbb{Q}} mA$ for some integer $m > 0$ and $A \in \mathrm{Cl}(S)$.
- Similar analysis for each class.
  When S has singularity of $2A_1 + A_3$, we have $(-K_S)^2 = 4$ and $\mathrm{Cl}(S)/\mathrm{Pic}(S) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
  $\Rightarrow 4A \in \mathrm{Pic}(S)$ and $(-K_S)^2 = \frac{m^2}{4^2}(4A)^2$.
  $\Rightarrow m^2 = \frac{4 \cdot 16}{(4A)^2}$ and $(4A)^2 \in \mathbb{Z}$ since $4A$ is Cartier.
  $\Rightarrow m \leq 6$ or $m = 8$.

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- only need to show that m cannot be 8.
  Let $p : Y \to S$ be the minimal resolution of S.
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Sketch of proof:

- there is a contraction $\pi : Z \to S$, where S is a canonical del Pezzo surfaces with Picard rank one or two (Miyanishi, Zhang).
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Terminal singularities

Lower dimensions

- $n = 3$, $X = \mathbb{P}^3(7, 5, 3, 2)$ has Fano index 17. It is the second largest Fano index for all $\mathbb{Q}-$Fano threefolds. $\mathrm{FI}(\mathbb{P}^3(7, 5, 4, 3)) = 19$ (Prokhorov).
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Theorem (Wang2023)

For each integer $n \geq 3$, let

- $a_0 = \frac{1}{2}(s_{n-1} - 1) - 1$,
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Then $X = \mathbb{P}^n(a_n, \ldots, a_0)$ is well-formed with terminal singularities and with Fano index $\frac{1}{2}(s_{n-1} - 1)^2 - 1$. In particular, $\mathrm{FI}(X) > 2^{2^{n-1}}$.

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Gorenstein

Theorem

For each integer $n \geq 1$, let $h = s_n - 1$.
Then $X = \mathbb{P}^n(h/s_0, \ldots, h/s_{n-1}, 1)$ is well-formed with Gorenstein canonical singularities and with Fano index h.

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Conjecture.

WPS $X = \mathbb{P}^N(a_0, \ldots, a_N)$ is a toric variety. In order to show X is canonical (or terminal), it is enough to check that each coordinate point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ is canonical (or terminal).

- the torus $T = (\mathbb{G}_m)^{N+1}/\mathbb{G}_m \cong (\mathbb{G}_m)^N$ acts on X by scaling the variables,
- The locus where X is canonical (or terminal) is open and T-invariant. Thus if X is canonical (or terminal) at a point q, then X is also canonical (or terminal) at all points p such that q is in the closure of the T-orbit of p.

There are two tricks originated from Reid-Tai criterion to check a quotient singularitiy is canonical or terminal.
Let $\frac{1}{r}(b_1, \ldots, b_s)$ be a well-formed quotient singularity

Lemma (ETW2021)

If some nonempty subset $I \subset \{b_1, ..., b_s\}$ has sum congruent to $0 \bmod r$ and $\gcd(I \cup \{r\}) = 1$, then the singularity is canonical.

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Let X be a Fano variety of dimension n. Define:

$$\mathrm{vol}(X) := \lim_{\ell \to \infty} h^0(X, -\ell K_X)/(\ell^n/n!)$$

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- Among all $n-$dimensional canonical toric Fano varieties for $n \geq 4$,
  $\mathbb{P}^n(1, 1, 2(s_n - 1)/s_{n-1}, \dots, 2(s_n - 1)/s_1)$has the largest volume $2(s_n - 1)^2$. (Balletti, Kasprzyk, and Nill)
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Gorenstein terminal

(Kasprzyk)
Odd dimensions:

- $\mathbb{P}^5(4, 3, 2, 1, 1, 1)$, volume 10368,
- $\mathbb{P}^7(28, 21, 14, 12, 6, 1, 1, 1)$, volume 49787136,
- $\mathbb{P}^9(1204, 903, 602, 516, 258, 84, 42, 1, 1, 1)$ volume 340424620687872.

They are the largest volume among all Gorenstin terminal WPS in dimension $n = 5, 7, 9$.

generalize to higher dimensions

For each odd integer $n = 2k + 1 \geq 5$, where integer $k \geq 2$, let

- $h = 2s_0 s_1 \cdots s_{k-1} = 2(s_k - 1)$,
- $a_0 = a_1 = a_2 = 1$,
- $a_{2i-1} = \frac{h}{2s_{k+1-i}} = s_0 s_1 \cdots \widehat{s_{k+1-i}} \cdots s_{k-1}$ for $2 \leq i \leq k-1$ when $k \geq 3$,
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generalize to higher dimensions

For each odd integer $n = 2k+1 \geq 5$, where integer $k \geq 2$, let

- $h = 2s_0 s_1 \cdots s_{k-1} = 2(s_k - 1)$,
- $a_0 = a_1 = a_2 = 1$,
- $a_{2i-1} = \frac{h}{2s_{k+1-i}} = s_0 s_1 \cdots \widehat{s_{k+1-i}} \cdots s_{k-1}$ for $2 \leq i \leq k-1$ when $k \geq 3$,
- $a_{2i} = \frac{h}{s_{k+1-i}} = 2s_0 s_1 \cdots \widehat{s_{k+1-i}} \cdots s_{k-1}$ for $2 \leq i \leq k-1$ when $k \geq 3$,
- $a_{n-2} = h/6 = s_0 s_2 \cdots s_{k-1}$,
- $a_{n-1} = h/4 = s_1 s_2 \cdots s_{k-1}$,
- $a_n = h/3 = 2s_0 s_2 \cdots s_{k-1}$.

Theorem (Wang2023)

Then Gorenstein terminal WPS $X = \mathbb{P}^n(a_n, \ldots, a_0)$ *has volume* $(-K_X)^n = 2^{\frac{n+1}{2}}(s_{\frac{n-1}{2}} - 1)^4$.

Conjecture: it has the largest volume among all Fano $n-$folds ($n \geq 5$ odd) with Gorenstein terminal singularities.

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Thank you!