Toric degenerations of partial flag varieties via matching fields and combinatorial mutations

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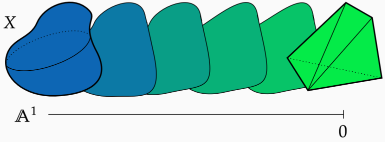

joint work with Oliver Clarke and Fatemeh Mohammadi

Outline

- Preliminaries
  - Toric degeneration
  - Grassmannian and flag varieties
- Toric degenerations from tropical geometry
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  - Tropicalization
- Matching fields and combinatorial mutations
  - Matching field polytopes
  - Combinatorial equivalence of the matching field polytopes
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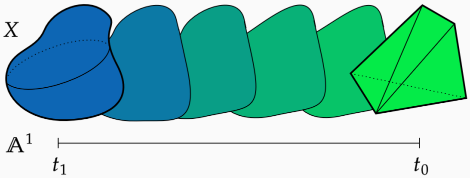
Preliminaries

Toric degenerations

A **toric degeneration** of a variety X is a flat family $\mathcal{F} \to \mathbb{A}^1$ such that:

- the fiber $\mathcal{F}_t$ over $t \in \mathbb{A}^1 \backslash \{0\}$ is isomorphic to X;
- the fiber $\mathcal{F}_0$ over 0 is a toric variety.



- Toric degenerations have been studied in algebraic geometry, representation theory, cluster algebra, and tropical geometry.
- The geometric invariants of X can be read from any fiber in the degeneration, in particular from the toric fiber.

Grassmannian and flag varieties

- The **Grassmannian** $\mathrm{Gr}(k,n)$ is the variety of k-dimensional linear subspaces in $\mathbb{K}^n$.
- The **flag variety** $\mathcal{F}\ell_n$ is the variety of flags $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n$, where $V_k \in \mathrm{Gr}(k,n)$. The flag variety naturally lives in a product of Grassmannians:

$$\mathcal{F}\ell_n \subseteq \mathrm{Gr}(1,n) \times \mathrm{Gr}(2,n) \times \cdots \times \mathrm{Gr}(n-1,n).$$

- The **partial flag variety** $\mathcal{F}\ell_n(\mathcal{I})$, with $[n] \supset \mathcal{I} = \{i_1 < i_2 < \cdots < i_k\}$, is the variety of flags $V_{i_1} \subsetneq V_{i_2} \subsetneq \cdots \subsetneq V_{i_q}$, where $V_{i_j} \in \mathrm{Gr}(i_j, n)$. The partial flag variety lives in a product of Grassmannian:

$$\mathcal{F}\ell_n(I) \subseteq \mathrm{Gr}(i_1,n) \times \mathrm{Gr}(i_2,n) \times \cdots \times \mathrm{Gr}(i_k,n).$$

Plücker variables

- $\mathrm{Gr}(k,n)$ can be embedded in a projective space via the **Plücker coordinates**:

$$\mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$$

where coordinates of $\mathbb{P}^{\binom{n}{k}-1}$ are labeled by k-subsets of $[n]$.

$$p_I = \det X[I] \text{ for } I \in \binom{[n]}{k}$$

- $\mathcal{F}\ell_n(\mathcal{I})$ can be embedded into a product of projective spaces $\mathbb{P}^{\binom{n}{i_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{i_k}-1}$, where coordinates are labeled by subsets of $[n]$:

$$p_I = \det X[I] \text{ for } I \subseteq [n], |I| \in \mathcal{I}$$

Toric degenerations from tropical geometry

- A classical way is via **Gröbner degenerations**.
- Let $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal. Given $w \in \mathbb{R}^{n+1}$ we can define the ideal

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle$$

where

$$\mathrm{in}_w(f) = \sum_{\alpha \cdot w \text{ minimal}} f_\alpha x^\alpha.$$

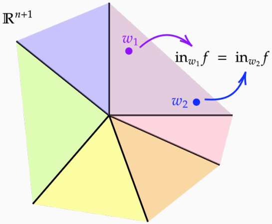
Example. Let
$f = p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \in \mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$. Then

  - for $w = (1, 0, 0, 0, 0, 1)$ we have $\mathrm{in}_w(f) = -p_{13}p_{24} + p_{14}p_{23}$;
  - for $w = (1, 1, 1, 2, 3, 4)$ we have $\mathrm{in}_w(f) = p_{14}p_{23}$.

- It is possible to generate a flat family of varieties over $\mathbb{A}^1$ such that the special fiber corresponds to the ideal $\mathrm{in}_w(I)$.
- If $\mathrm{in}_w(I)$ is a toric ideal, we have a toric degeneration.

Gröbner fan

- The **Gröbner fan** of $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is a fan in $\mathbb{R}^{n+1}$ where w_1 and w_2 lie in the same cone if and only if they give the same initial ideal.



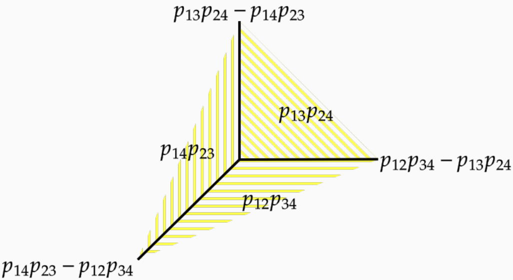
- Not every point in the Gröbner fan gives a toric degeneration: a generic weight $w \in \mathbb{R}^{n+1}$ give rise to a monomial ideal $\mathrm{in}_w(I)$.
- $\mathrm{in}_w(I)$ needs to be binomial and prime

$$\Downarrow$$

We restrict to the w in the fan such that $\mathrm{in}_w(I)$ contains no monomial.

Gröbner fan of $\mathrm{Gr}(2,4)$

Example. Consider $\mathrm{Gr}(2,4) = V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$. The Gröbner fan consists of 7 cones:



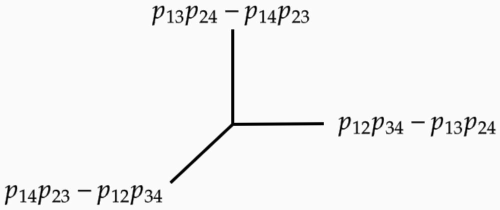
Idea: restrict to $\{w \in \mathbb{R}^{n+1} \mid \min\{\alpha \cdot w \mid f_\alpha \neq 0\}$ is achieved at least twice$\}$.

Tropicalization

This space is the **tropicalization** of $X = V(I)$:

$$\text{trop}(X) = \bigcap_{f \in I} \{w \in \mathbb{R}^{n+1} \mid \min\{\alpha \cdot w \mid f_\alpha \neq 0\} \text{ is achieved at least twice}\}$$

Example. For $\text{Gr}(2,4)$ we get 3 top-dimensional cones. All of them give rise to toric degenerations of $\text{Gr}(2,4)$.

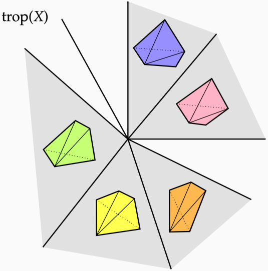


Tropicalization and toric degenerations

Moreover $\text{in}_w(I)$ needs to be binomial and prime.

$$\Downarrow$$

We restrict to the cones giving prime initial ideals, which we call prime cones.



Tropicalization of Grassmannian and flag varieties

Computing points in top-dimensional cones of the tropicalization of a variety is not trivial:

- $\text{trop}(\text{Gr}(3,6))$ is a 3-dimensional fan with 1005 maximal cones. They merge into 7 symmetry classes, 6 of which give non-isomorphic toric degenerations.
- $\text{trop}(\text{Gr}(3,7))$ is a 5-dimensional fan with 252000 maximal cones. They merge into 125 cones modulo S_7, 69 of which give non-isomorphic toric degenerations.
- $\text{trop}(\mathcal{F}\ell_5)$ has 69780 maximal cones, 536 modulo the action of $S_5 \times \mathbb{Z}_2$. 180 give toric degenerations.

Matching fields and combinatorial mutations

Matching fields

We want ways to generate points in the tropicalization of these varieties.

A **matching field for** $\mathrm{Gr}(k,n)$ is a map

$$\Lambda : \binom{[n]}{k} \to S_k.$$

A **matching field for** $\mathcal{F}\ell_n(\mathcal{I})$ is a map

$$\Lambda : \{I \subset [n] \mid |I| \in \mathcal{I}\} \to \bigsqcup_{k \in \mathcal{I}} S_k.$$

A matching field is **coherent** is there exists a matrix $M \in \mathbb{R}^{(n-1)\times n}$ such that for every $I \subset [n]$, $|I| = k$

$$\Lambda(I) = \mathrm{argmin}_{\sigma \in S_k} \sum_{i=1}^{k} M_{i,\sigma(i)}$$

and the minimum is attained at a unique $\sigma \in S_k$.

Matching field weight and polytope

Fix a coherent matching field Λ for $\mathcal{F}\ell_n(\mathcal{I})$. We associate:

- the **weight vector** w_Λ

$$w_\Lambda = \left(\min_{\sigma \in S_k} \sum_{i=1}^{k} (M_\Lambda)_{i,\sigma(i)} \right)_{I \subset [n], |I| = k \in \mathcal{I}}$$

- If Λ is a matching field for $\mathrm{Gr}(k,n)$, the **polytope** P_Λ^k is

$$P_\Lambda^k = \mathrm{conv}\left(E_\sigma \mid \sigma = \Lambda(I) \text{ for some } I \in \binom{[n]}{k} \right)$$

where $(E_\sigma)_{ij} = \begin{cases} 1 \text{ if } j = \sigma(i) \\ 0 \text{ otherwise.} \end{cases}$

- For a partial flag variety $\mathcal{F}\ell_n^{\mathcal{I}}$:

$$P_\Lambda^{\mathcal{I}} = P_\Lambda^{i_1} + \cdots + P_\Lambda^{i_k}$$

- **Proposition.** The matching field polytope P_Λ is normal.

A matching field polytope for $\mathrm{Gr}(2,4)$

Consider the matching field $\Lambda : \binom{[4]}{2} \to S_2$ defined by

$$M_\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

The weight vector is

$$w_\Lambda = (2,3,1,2,1,1)$$

and the polytope is given by

$$P_\Lambda = \mathrm{conv}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

Note that

$$\mathrm{in}_w(I_{2,4}) = \mathrm{in}_w(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) = (p_{12}p_{34} + p_{14}p_{23}).$$

Matching fields and toric degenerations

- Not every matching field defines a toric degeneration: the ideal $\text{in}_{w_\Lambda}(I)$ might not be prime.
- There is a different way to construct a toric degeneration from a matching field. Λ defines a monomial map

$$\phi_\Lambda : \mathbb{C}[p_I] \to \mathbb{C}[x_{ij}]$$

  sending p_I to the monomial of the determinant of X_I corresponding to $\Lambda(I)$.
- $\ker(\phi_\Lambda)$ is a toric ideal, i.e. it is binomial and prime. It is possible to prove that

$$\text{in}_{w_\Lambda}(I) \subseteq \ker(\phi_\Lambda)$$

- The toric variety defined by $\ker(\phi_\Lambda)$ is

$$\mathbb{C}[p_I]/\ker(\phi_\Lambda) = \mathbb{C}[\mathrm{Cone}(P_\Lambda) \cap \mathbb{Z}^m \times \mathbb{Z}].$$

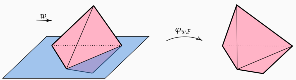
Matching fields and toric degenerations

Theorem 1. Let Λ be a matching field for $\mathcal{F}\ell_n(J)$. If P_Λ is combinatorial mutation equivalent to the Gelfand-Tsetlin polytope, then Λ gives rise to a toric degeneration of $\mathcal{F}\ell_n(J)$.

Let N be a lattice and $M = N^*$. Let $w \in M$ be a primitive vector and $F \subseteq w^\perp \subset N_\mathbb{R}$ a lattice polytope. The tropical map

$$\varphi_{w,F} : M_\mathbb{R} \to M_\mathbb{R}, \quad x \mapsto x - u_{\min}(x)w$$

$u_{\min} = \min\{\langle x, f\rangle \mid f \in F\}$, is a **combinatorial mutation** of a lattice polytope $P \subset M_\mathbb{R}$ if $\varphi_{w,F}(P)$ is convex.



The **Gelfand-Tsetlin polytope** is the polytope P_{GT} associated to the matching field Λ_{GT}:

$$\Lambda_{GT} : I \mapsto \text{id}$$

Proof of theorem 1

Theorem 1. Let Λ be a matching field for $\mathcal{F}\ell_n(J)$. If P_Λ is combinatorial mutation equivalent to the Gelfand-Tsetlin polytope, then Λ gives rise to a toric degeneration of $\mathcal{F}\ell_n(J)$.

Idea of proof.

- Hilbert function of $\mathcal{F}\ell_n(J)$ = Hilbert function of $\mathbb{C}[P_I]/\operatorname{in}_{GT}(I) =$
  $=$ Ehrhart polynomial of P_{GT} = Ehrhart polynomial of P_Λ =
  $=$ Hilbert function of $\mathbb{C}[P_I]/\ker(\phi_\Lambda)$.
- Hilbert function of $\mathcal{F}\ell_n(J)$ = Hilbert function of $\mathbb{C}[P_I]/\operatorname{in}_{w_\Lambda}(I)$.
  Then $\operatorname{in}_{w_\Lambda}(I) = \ker(\phi_\Lambda)$, in particular

$$\operatorname{in}_{w_\Lambda}(I) \text{ is toric.}$$

A family of matching fields

Goal. Define a large family of matching fields and prove that the corresponding polytopes are combinatorial mutation equivalent to the GT polytope.

Let $\sigma \in S_n$ and consider the matching field Λ_σ associated to the matrix:

$$M^\sigma = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ Nn & N(n-1) & \dots & N \\ \vdots & \vdots & & \vdots \\ N^{k-2}(n-1) & N^{k-2}(n-1) & \dots & N^{k-2} \end{pmatrix}$$

for $N \geq n+1$.

Note that with this notation, the Gelfand-Tsetlin polytope is associated to the permutation $w_0 = (n\ n-1\ \dots\ 2\ 1)$.

Matching field polytopes are mutation equivalent

Theorem 2. If $\sigma \in S_n$ is a permutation that avoids the pattern 4123, 3124, 1423 and 1324, then the polytope P_σ^J associated to the matching field Λ_σ for the partial flag variety $\mathcal{F}\ell_n(J)$, is combinatorial mutation equivalent to the Gelfand-Tsetlin polytope $P_{GT} = P_{w_0}$.

Idea of proof.

- We just need to prove it for Grassmannians.
- We construct combinatorial mutations

$$P_\sigma \to P_{\sigma_1} \to P_{\sigma_2} \to \cdots \to P_{\sigma_m} = P_{(n\ n-1\ \dots\ 1)} = P_{GT}$$

where $\sigma_{i+1} = (\ell\,\ell+1)\sigma_i$.

Example. Consider $\sigma = (6\ 2\ 3\ 5\ 4\ 1)$. Then the combinatorial mutations from P_σ to P_{w_0} will follow the sequence:

$(6\ 2\ 3\ 5\ 4\ 1) \to (6\ 2\ 4\ 5\ 3\ 1) \to (6\ 3\ 4\ 5\ 2\ 1) \to (6\ 3\ 5\ 4\ 2\ 1) \to (6\ 4\ 5\ 3\ 2\ 1) \to w_0$

Example of combinatorial mutation

Consider P_σ with $\sigma = (6\,2\,4\,3\,5\,1)$ for $\mathrm{Gr}(3,6)$. We want to construct a (sequence of) combinatorial mutations to P_τ where $\tau = (6\,2\,5\,3\,4\,1)$.

It is possible to construct w_1, w_2 and F_1, F_2 such that

$$P_\tau = \varphi_{-w_\mathbf{1}, F_\mathbf{1}} \circ \varphi_{w_\mathbf{2}, F_\mathbf{2}} \circ \varphi_{w_\mathbf{1}, F_\mathbf{1}}(P_\sigma).$$

The polytope $Q = \varphi_{w_\mathbf{2}, F_\mathbf{2}} \circ \varphi_{w_\mathbf{1}, F_\mathbf{1}}(P_\sigma)$ is not a lattice polytope. It corresponds to the non-prime cone in $\mathrm{Gr}(3,6)$.

- Can we generalize this construction to other permutations containing forbidden patterns? Can we generalize it to higher Grassmannians?

Computational results

We can generalize the matching fields M_σ to M_σ^c by multiplying the second row by c.

Considering the matching fields M_σ^c we get:

- for $\mathrm{Gr}(3,6)$ all the 6 possible toric degenerations.
- for $\mathrm{Gr}(3,7)$, 40 out of 69 possible toric degenerations.
- for $\mathcal{F}\ell_4$ all the 4 possible toric degenerations.
- for $\mathcal{F}\ell_5$, 22 out of 180 possible toric degenerations.

Oliver Clarke, Fatemeh Mohammadi, Francesca Zaffalon
Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes

Open problems

- Do combinatorial mutation always move points from one cone to an adjacent one?
- Do we only move along facets of one maximal cone via combinatorial mutations?
- How can we describe the remaining toric degenerations?

Thank you!