Moduli of curves & K-stability /$\mathbb{C}$

<u>Goal</u>: understand different compactifications (or birational models) of M_g using K-stability / K-moduli theory

<u>Strategy</u>: for a general $C \in M_g$, $C \xhookrightarrow{|\omega_C|} X$ for some surface X, study pairs (X, C)

<u>A perfect model</u>: $g = 3$ $C \hookrightarrow \mathbb{P}^2$ as a non-hyperelliptic plane quartic

$(\mathbb{P}^2, cC_4)$ is log Fano if $0 < c < \frac{3}{4}$ $\rightsquigarrow$ K-moduli

log CY if $c = \frac{3}{4}$ $\rightsquigarrow$ *moduli of log CY

log gen'l type if $\frac{3}{4} < c \le 1$ $\rightsquigarrow$ KSBA moduli

<u>Hassett-Keel program</u> for $\overline{M}_g$:

$0 \le \alpha \le 1$ s.t. $K_{\overline{M}_g} + \alpha \Delta$ is ps-eff. $\Delta := \sum_{i=0}^{\lfloor g/2 \rfloor} \Delta_i$

$$\rightsquigarrow \overline{M}_g(\alpha) := \mathrm{Proj}\, R(K_{\overline{M}_g} + \alpha\Delta)$$

[Hassett-Hyeon]

$$\begin{tikzcd}
\overline{M}_3(1) = \overline{M}_3 \arrow[dd, phantom, "\wr"] \arrow[dr] & & & \overline{M}_3(\tfrac{7}{10}-\varepsilon) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wr"] & \\
& \overline{M}_3(\tfrac{9}{11}) \arrow[r] \arrow[dd, phantom, "\wr"] & \overline{M}_3(\tfrac{7}{10}) \arrow[dd, phantom, "\wr"] & & \overline{M}_3(\tfrac{17}{28}) \arrow[d, phantom, "\wr"] \\
\overline{M}^{KSBA}(1) \arrow[dr] & & & \overline{M}^{K}(\tfrac{3}{4}-\varepsilon) \arrow[dl] \arrow[dr] & \overline{M}^{GIT} \arrow[d, phantom, "\wr"] \\
& \overline{M}^{KSBA}(\tfrac{3}{4}+\varepsilon) \arrow[r] & \overline{M}^{CY}(\tfrac{3}{4}) & & \overline{M}^{K}(\tfrac{3}{8})
\end{tikzcd}$$

Hassett

[ADL]

Hacking

[ABBDILW]

$- \; g = 4$ $\qquad C \in M_4$ gen'l

$C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. $\qquad (\mathbb{P}^1 \times \mathbb{P}^1, cC)$

Today: $g = 6$

Geometry of genus 6 curves

$C \in M_6$ gen'l

Brill-Noether: $\dim g^2_6 = 0$

$C \xrightarrow{\varphi} \mathbb{P}^2$ sextic curve. 4 nodes

$\varphi(C)$ = irred. sextic w/ 4 nodes — in gen'l positition / NOT

C is special — plane quintic / hyperelliptic / bielliptic (2:1 map to elliptic) / trigonal (3:1 to $\mathbb{P}^1$)

$\Updownarrow$

$\dim g^2_6 > 0$

blow-up 4 nodes of $\varphi(C) \subset \mathbb{P}^2$

$\rightsquigarrow \mathrm{Bl}_{p_1 - p_4} \mathbb{P}^2$ = sm del Pezzo of deg = 5 (unique)

Σ =

$C \hookrightarrow \Sigma$. $\quad C \in |-2K_\Sigma|$. $\quad \mathrm{Aut}(\Sigma) = \mathfrak{S}_5$

$\rightsquigarrow$ study the pair (Σ, cC)

§ K-stability

<u>Valuative criterion</u>: (Fujita, Li)

(X, D) log Fano. $\quad E$: prime div. $/X \subset \tilde{X} \xrightarrow[\text{bir.}]{\pi} X$

$$A_{(X,D)}(E) = 1 + \mathrm{ord}_E\left(K_{\tilde{X}} - \pi^*(K_X + D)\right)$$

$$S_{(X,D)}(E) = \frac{1}{(-K_X-D)^n} \int_0^\infty \mathrm{vol}(-\pi^*(K_X+D) - tE)\, dt$$

$$\beta(E) := A(E) - S(E)$$

klt. $-K_X - D$ ample

<u>Thm/Defn</u>: (X, D) log Fano, it is

- K-semistable $\iff$ $\beta(E) \geq 0$, $\forall E/X$

- K-stable $\Leftrightarrow$ $>$

<u>Examples</u>: 1) $\dim X = 2$. K-ss $\Leftrightarrow$ $X \neq Bl_p \mathbb{P}^2$ or $Bl_{p,q}\mathbb{P}^2$

2) $\dim X = 3$. 105 families $\sim 70^+$ of them have K-ss members

<u>K-moduli Theorem</u>: (Weak version) $\forall\, c \in (0, \frac{1}{2})$ rat'l.

$\exists$ a proj. scheme $\overline{M}^K(c)$ parametrizing K-ps pairs (X, cD) which admit a $\mathbb{Q}$-Gorenstein smoothing to (Σ, cC), $C \overset{\text{sm.}}{\in} |-2K_\Sigma|$

<u>Thm</u>: (ADL'19) $\exists$ $0 = c_0 < c_1 < \cdots < c_\ell = \frac{1}{2}$ (~ walls) s.t.

1) $c \in (c_i, c_{i+1})$. $\overline{M}^K(c)$ is indep. of c

2) $c = c_i$:

$$\begin{array}{ccccc} & \overline{M}^K(c_i-\varepsilon) & \overset{\text{bir.}}{\dashrightarrow} & \overline{M}^K(c_i+\varepsilon) & \\ E_- \subset & \searrow \varphi_- & & \swarrow \varphi_+ & \supset E_+ \end{array}$$

$$Z \subseteq \overline{M}^K(c_i)$$

$$\dim E_- + \dim E_+ = \dim Z + \dim M - 1$$

Two models of $\overline{M}_6$:

1) $|-2K_\Sigma| / B_5 \overset{\text{bir.}}{\dashrightarrow} \overline{M}_6$

2) (Σ, C). double cover of Σ branched along C

$\rightsquigarrow Y$: K3. $\mathcal{F}^* = \overline{(D/\Gamma)}^{bb}$: Baily-Borel comp.

<u>Thm A</u> (Z' 22) $\overline{M}^K(c) = \overline{\{(\Sigma, cC)\}}$. $C \in |-2K_\Sigma|$

1) For $c = \varepsilon$. $\overline{M}^K(c) \simeq |-2K_\Sigma| / B_5$. $\rho(\overline{M}^K(\varepsilon)) = 1$

2) For $c = \frac{1}{2} - \varepsilon$. $\overline{M}^K(c) \to \mathcal{F}^*$ (isom. in codim $= 1$)

is ample model of Hodge $\mathbb{Q}$-l.b.

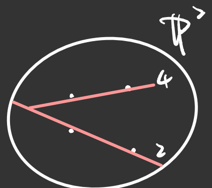
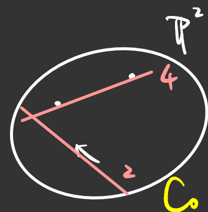
3) Among all the walls, $\exists$ 3 $c_i \in \{\frac{1}{7}, \frac{1}{52}, \frac{1}{4}\}$

s.t. $\varphi_{i+} : \overline{M}^K(c_i + \varepsilon) \to \overline{M}^K(c_i)$ is a div.

contraction. $\rho(\bar{M}^K(\tfrac{1}{2}-\varepsilon)) = 4$

4) all ADE dP^5 appear in $\bar{M}^K(c)$ for some c

<u>Example</u>: (First wall $c = \frac{1}{17}$)

$c = \frac{1}{17} - \varepsilon$	$c = \frac{1}{17}$	$c = \frac{1}{17} + \varepsilon$
$E_- = \{$  $\}$, $E_- = \{pt\}$	 $= Z$	$\{(\Sigma_1, C) \mid C \in \lvert -2K_{\Sigma_1} \rvert,\ p \notin C,\ C \neq C_0\} = E_+$

$\Sigma_1 := dP^5$ w/ an A_1-sing. p

§ Maps to $\bar{M}_6$

$\bar{M}^K(\varepsilon) \dashrightarrow M_6$ Thm A 1) $\Rightarrow$ image = locus of sm. genus 6 curve w/ 5 g^2_6

$\bar{M}^K(\tfrac{1}{2}-\varepsilon) \dashrightarrow M_6$ 2) + 3) image contains trigonal curves + plane quintic curve

4) image contains all the BN-gen'l curves

Thm B: Image of $\overline{M}^K(\tfrac{1}{2}-\varepsilon) \dashrightarrow M_6$ only misses loci of

hyperelliptic & bielliptic curves.

KSBA moduli.

$(\Sigma, c(2C_3))$ $\quad \mathrm{lct} = \frac{1}{2}$

$c > \frac{1}{2}$.

Hassett-Keel: $\overline{M}_g(\alpha) = \mathrm{Proj}\, R(K_{\overline{M}_g} + \alpha\Delta)$ $\qquad \alpha_i$'s

$$\begin{tikzcd}
\overline{M}_g(\alpha_i+\varepsilon) \arrow[dr] & & \overline{M}_g(\alpha_i-\varepsilon) \arrow[dl] \\
& \overline{M}_g(\alpha_i) &
\end{tikzcd}$$

Known: 1) $\alpha = \frac{9}{11}, \frac{7}{10}, \frac{2}{3}$ first 3 walls

2) $g=2$ Hassett

3) $g=3$ Hyeon – Lee

4) $g=4$ miss $\frac{5}{9} \leq \alpha \leq \frac{2}{3}$ [CMJL]

5) $g = 5, 6$ last models

FS (g=5) Müller (g=6)

<u>Thm C</u> $(Z' \geq 3)$ $0 \leq c \leq \frac{11}{52}$. $\alpha(c) = \frac{32 - 19c}{94 - 68c}$. Then

$$\overline{M}^K(c) \simeq \overline{M}_6(\alpha(c)).$$ In particular.

last six walls / log can. models $= \left\{ \frac{16}{47}, \dots, \frac{47}{134} \right\}$

(Σ, cC). $0 \leq c \leq 1$

$$\begin{tikzcd}
\overline{M}^{KSBA}(1) \arrow[d, "\text{adjunction}"] & \mathcal{F}^* \arrow[l, squiggly, "?"'] & \overline{M}^K(\varepsilon) \arrow[l, squiggly, "\checkmark"'] \arrow[r] \arrow[d, "\wr"] & \overline{M}^K(0) = \{pt\} \arrow[d, "\wr"] \\
\overline{M}_6 = \overline{M}_6(1) & & \overline{M}_6(\frac{16}{47} + \varepsilon) \arrow[ll, squiggly, "?"'] \arrow[r] & \overline{M}_6(\frac{16}{47})
\end{tikzcd}$$