Wall crossing for K-stability with multiple boundaries ($/\mathbb{C}$)

MMP $\Rightarrow$ General type, K_X ample; Calabi-Yau, $K_X \equiv 0$

Fano

$-K_X$ ample

General type: KSBA-moduli (Kollár ...)

Calabi-Yau: Hodge theory $\rightsquigarrow$ K3

GIT $\rightsquigarrow$ CY hypersurface

K-stability (X, D)

$\uparrow$

Fano

MMP

Fano: K-moduli (Xu ...)

I K-stability and K-moduli

(X, Δ) Fano if klt, $-(K_X+\Delta)$ ample

Def (Fujita, Li)

For a log Fano (X, Δ)

it is k.s.s $\overset{\Delta}{\Longleftrightarrow}$ $\beta_{X,\Delta}(E) > 0$ for any prime div E/X

- E/X: E is a prim div on a birational model: $\mu: Y \to X$, $Y \supset E$

- $\beta_{X,\Delta}(E) := A_{X,\Delta}(E) - S_{X,\Delta}(E)$

$$A_{X,\Delta}(E) = \mathrm{ord}_E(K_Y - f^*(K_X+\Delta)) + 1$$

$$S_{X,\Delta}(E) = \frac{1}{\underbrace{(-K_X-\Delta)^d}} \int \underbrace{\mathrm{vol}(-K_X-\Delta-tE)}\, dt$$

Rk $$\delta(X,\Delta) := \inf_{E/X} \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}$$

(X,Δ) is K-ss $\Leftrightarrow$ $\delta(X,\Delta) \geq 1$

K-moduli Thm (X_u . . .)

Fix $\underbrace{d, v, I}$ $\leftarrow$ finite set of non-negative rational numbers

Then, there exists a Artin Stack of finite type, $\mathcal{M}^K_{d,v,I}$, parametrizing ly Fano (X, Δ) with

$$\begin{cases} \dim X = d \\ \Delta \in I \\ (-(K_X+\Delta))^d = v \\ (X, \Delta) \text{ is K-ss} \end{cases}$$

Moreover, $\mathcal{M}^K_{d,v,I} \longrightarrow M^K_{d,v,I}$

Good moduli space

projective scheme.

parametrize K-poly stable objects

Rk

- Define the K-moduli functor
- Confirm various moduli property

e.g. boundedness, separatedness, properness, projectivity

Thm (Jiang)

$$\left\{ X \;\middle|\; \begin{array}{l} \text{Fano of dim } d \\ (-K_X)^d = v \\ \delta(X) \geq a > 0 \end{array} \right\} \text{ is bdd}$$

bdd: $\mathcal{X} \xrightarrow{f} T$ finite type

II Wall crossing for K-stability

$$\left\{ (X, D) \;\middle|\; \begin{array}{l} X \text{ Fano of dim } d \\ \boxed{D \sim_{\mathbb{Q}} -K_X}, \; D \in I \\ (-K_X)^d = v \\ (X, cD) \text{ is K-ss for some} \end{array} \right\} =: \mathcal{F}$$

$c \in [0, 1)$

(X, cD) : The K-ss may change as we vary c

e.g * $(\mathbb{P}^2, cQ)$ is K-ss for $c \in [0, \frac{3}{4}]$

Smooth Conic

not K-ss for $c \in (\frac{3}{4}, 1)$

* $(\mathbb{P}^n, c S_d)$ is K-ss for

$d \leq n$ $\quad c \in \left[0, \frac{(n+1)(d-1)}{nd}\right]$

not K-ss for $\left(\frac{(n+1)(d-1)}{nd} \quad 1\right)$

* if X is K-ss Fano &

$(X \; D)$ is log canonical,

then (X, cD) is k.ss for $c \in [0,1)$

$\forall$ $\beta_{X, cD}(E) = A_X(E) - c\,\mathrm{ord}_E D - (1-c) S_X(E)$

Interpolation property for K-stability:

if $(X, c_1 D)$ $(X, c_2 D)$ are k.ss

then (X, cD) is k.ss for $c \in [c_1, c_2]$

Def

$\underline{Kss(X, D)} := \overline{\{c \in [0,1) \mid (X, cD) \text{ is k.s.s}\}}$

Thm $(A$-D-$L, -)$ (X, D)

$\mathcal{F}$: d, v, I

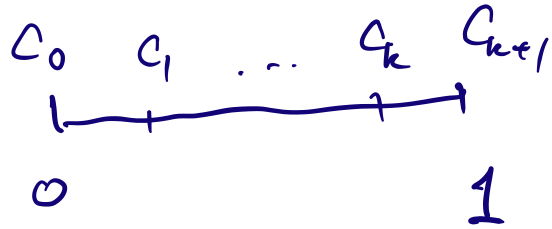
There exists finitely many rational numbers

$$0 = c_0 < c_1 < \cdots < c_k < c_{k+1} = 1$$

s.t $\forall\ (X, D) \in \mathcal{F}$

$$\mathrm{Kss}(X, D) = [c_i, c_j] \quad \text{for some } 0 \le i \le j \le k+1$$

In other words, we have a finite chamber decomposition for



$[c_1, c_2]$

Idea of pf

boundedness of $\mathcal{F}$

1 - Gap property for K-stability:

$\exists\ \varepsilon(d, v, I) > 0$ s.t.

$\forall\ (X, D) \in \mathcal{J}$, (X, cD) is K.ss

for $\underbrace{c \leq 1 - \varepsilon(d, v, I)}$

(X, cD) is K.ss $\Longrightarrow$

$$\frac{A_{X,cD}(E)}{S_{X,cD}(E)} = \frac{A_X(E) - c\,\mathrm{ord}_E D}{(1-c)\,S_X(E)}$$

$$\Longrightarrow \quad \frac{A_X(E)}{(1-c)\,S_X(E)} \geq 1$$

$$\Longrightarrow \quad \frac{A_X(E)}{S_X(E)} \geq 1-c \geq \varepsilon(d, v, I)$$

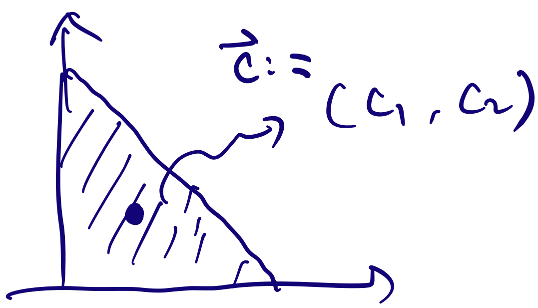
$\rightsquigarrow \delta(X) \geq \varepsilon(d, v, I)$

$\rightsquigarrow$ badness $\#$

III multiple boundaries

$$\mathcal{E} := \left\{ \left(X, \sum_{j=1}^{k} D_j\right) \;\middle|\; \begin{array}{l} X \text{ Fano of dim } d \\ (-K_X)^d = v \\ D_j \sim_{\mathbb{Q}} -K_X, \; D_j \in I \\ \left(X, \sum_{j=1}^{k} c_j D_j\right) \text{ is K-ss Fano} \\ \text{for some } 0 \leq c_j < 1 \end{array} \right\}$$

Rk $\quad \sum_{j=1}^{h} c_j < 1$



$$\mathrm{kss}\left(X, \sum_{j=1}^{k} D_j\right) = \left\{ (c_1, \dots, c_k) \;\middle|\; \left(X, \sum_{j=1}^{k} c_j D_j\right) \text{ is K-ss} \right\}$$

Thm (—)

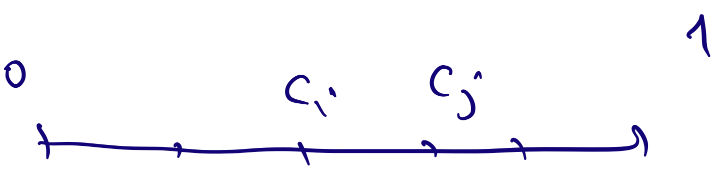
(1) $\forall \left(X, \sum_{j=1}^{k} D_j\right) \in \mathcal{E}$

$\mathrm{Kss}\left(X, \sum_{j=1}^{k} D_j\right)$ is a <u>rational polytope</u>

(2) $$\left\{ \mathrm{Kss}\left(X, \sum_{j=1}^{k} D_j\right) \,\middle|\, \left(X, \sum_{j=1}^{k} D_j\right) \in \mathcal{E} \right\}$$

is a finite set

<u>Rk</u>



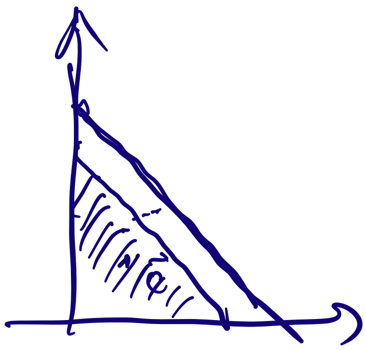
$[c_i . c_j]$

<u>Idea</u>

- polytope
- Finiteness

Finiteness : boundedness of $\mathcal{E}$.

1- Gap property for K-stability



$$(X, \sum_{j=1}^{k} D_j) \in \mathcal{E}$$

$(X, \sum c_j D_j)$ K-ss

$$\sum_{j=1}^{k} c_j \rightsquigarrow 1$$

$\exists\ \varepsilon(d, v, I) > 0$ s.t.

$(X, \sum_{j=1}^{k} c_j D_j)$ is K-ss with

$$\sum_{j=1}^{k} c_j \le 1 - \varepsilon(d, v, I)$$

$$\delta\Big(X, \sum_{j=1}^{k} c_j D_j\Big) \geq 1$$

$$\Rightarrow \frac{A_{X,\sum c_j D_j}(E)}{S_{X,\sum c_j D_j}(E)} = \frac{A_X(E) -}{(1-\sum c_j)\, S_X(E)} \geq 1$$

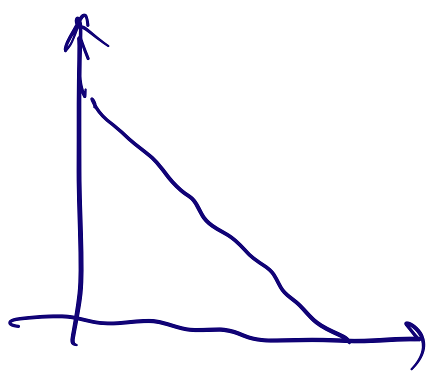
$$\Rightarrow \frac{A_X(E)}{S_X(E)} \geq 1 - \sum c_j \geq \varepsilon(d, v, I)$$

$$\Rightarrow \delta_X(E) \geq \varepsilon(d, v, I)$$

$\rightsquigarrow$ $\mathcal{E}$ is log bdd.

<u>polytope:</u>

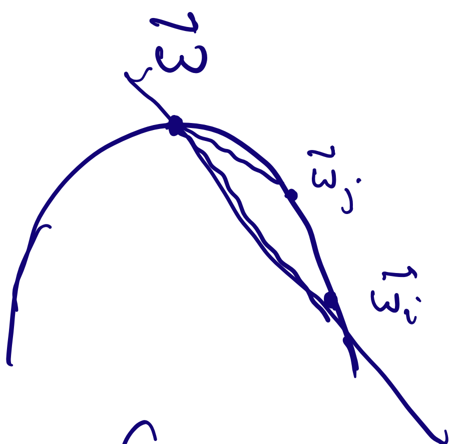
$$\Big(X, \sum_{j=1}^{2} D_j\Big) \in \mathcal{E}$$



$$\boxed{\beta_{X, x_1 D_1 + x_2 D_2}(E)} = A_X(E) - \sum_{j=1}^{2} x_j \operatorname{ord}_E D_j - \left(1 - \sum_{j=1}^{2} x_j\right) \delta_X(E)$$

$$\mathrm{Kss}\left(X, \sum_{j=1}^{2} D_j\right) = \bigcap_{E/X} \left\{ \beta_{X, x_1 D_1 + x_2 D_2}(E) \geq 0 \right\}$$

(Fujita - Li' criterion)



To prove it is polytope <u>locally at $\vec{\omega}$</u>

Suppose It is not locally polytope at $\vec{\omega}$

One could chose $\{\vec{\omega_i}\} \rightsquigarrow \vec{\omega}$

and $\overline{\vec{w}_i \vec{w}}$ separates KSB $(X, \sum_{j=1}^{2} D_j)$

for $i \gg 1$

$\vec{w}_i, \vec{w}$ (i.e on the same hyperplane

defined by β: $\exists$ E_i / X

s.t $\beta_{(X, \vec{w}\vec{D})}(E_i) = \beta_{X, \vec{w}_i \vec{D}}(E_i) = 0$

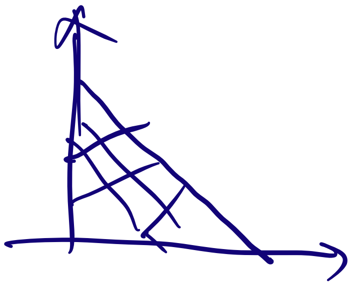
$\vec{w}, \vec{w}_i$ $\qquad \beta_{X, \sum x_j D_j}(E_i) \geq 0$.

Contradiction $\boxtimes$

Corollary

Fix k, d, v, I.

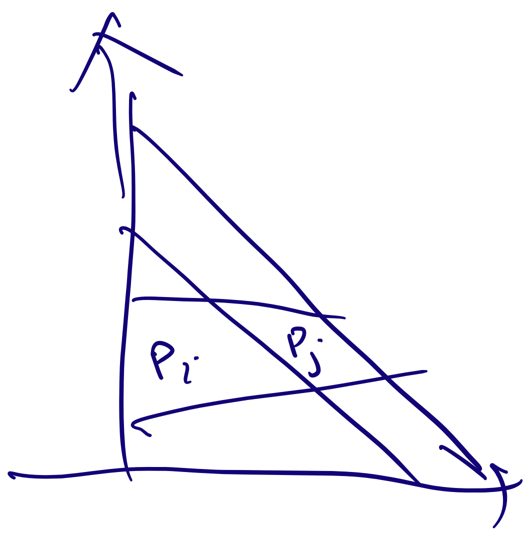
There exists a finite Chamber decomposition of



$$= \bigcup_{i=1}^{s} P_i$$

where P_i are rational polytopes and

$$P_i^\circ \cap P_j^\circ = \phi$$

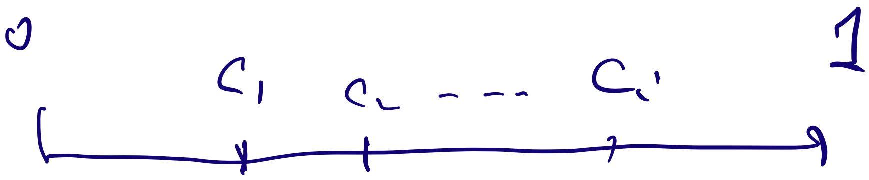


& k.i.s of $\left(X, \sum_{j=1}^{k} x_j D_j\right)$

does not change if we vary $\vec{x} = (x_1 \dots x_k) \in P_j$

based on this, we have a wall-crossing

for K-moduli,



$$\mathcal{M}^K_{d,v,I,c} \qquad (X, cD) \text{ where } (X,D) \in \mathcal{F}.$$

$$\begin{tikzcd}
\mathcal{M}^K_{d,v,I,c_i-\varepsilon} \arrow[r, hook] \arrow[d] & \mathcal{M}^K_{d,v,I,c_i} \arrow[d] & \mathcal{M}^K_{d,v,I,c_i+\varepsilon} \arrow[l, hook'] \arrow[d] \\
M^K_{d,v,I,c_i-\varepsilon} \arrow[r, "\varphi_i^-"] & M^K_{d,v,I,c_i} & M_{d,v,I,c_i+\varepsilon} \arrow[l, "\varphi_i^+"']
\end{tikzcd}$$

$(\varphi_i^-, \varphi_i^+)$ behaves like a flip

$$\left(\mathbb{P}^n, \ {}^{x}\mathcal{O} + {}^{y}L \right)$$

$$\left(\mathbb{P}^n, \ \mathcal{O}_1 + \mathcal{O}_2 \right)$$

K-ps

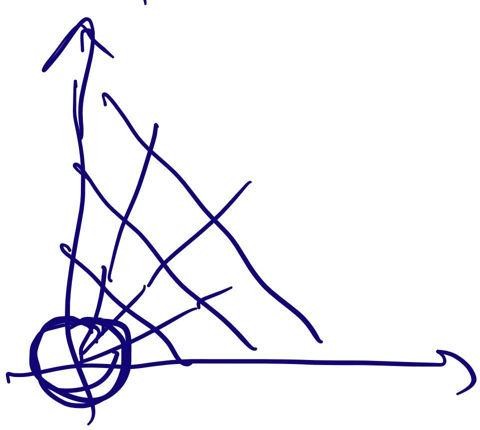
$$D \in \frac{1}{\ell} \left| -\ell K_X \right| =: \mathbb{P}$$

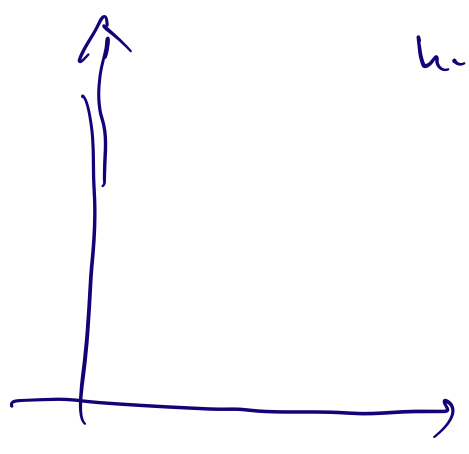
$\downarrow$

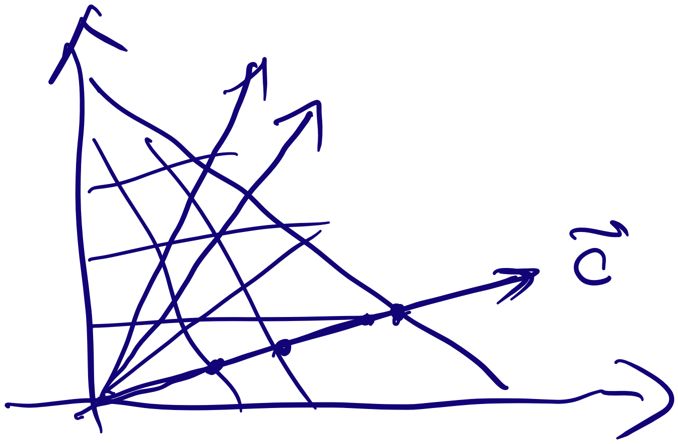
$$(X \quad \varepsilon D)$$

$\boxed{\text{K.ss of } (X, \varepsilon D)} \iff$ GIT. ss of D

w.r.t. $\mathrm{Aut}(X) \curvearrowright \mathbb{P}$



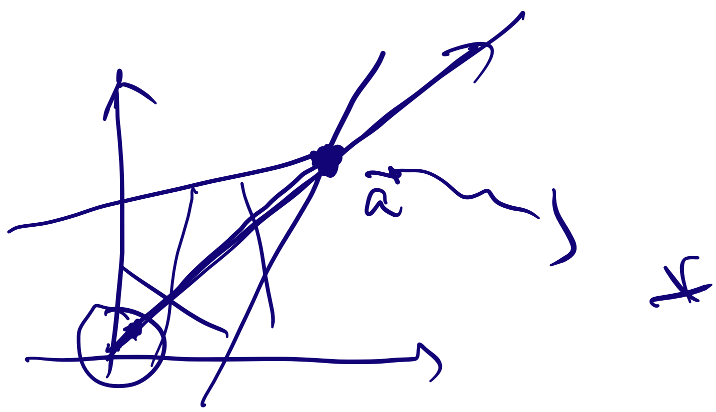




$(X, t\vec{C}\,\tilde{D})$

$\sum d_j < n+1$

$(\mathbb{P}^n, \sum c_j S_{d_j})$



$(\mathbb{P}^4 \quad Q_1+Q_2)$

$\mathcal{M}^k_{n,\vec{d},\vec{a}}$

$(\mathbb{P}^n, \sum a_j S_{d_j})$

$\boxed{M^{K}_{n,\vec{d}}}$

$\bigcap_{j} S_{d_j}$