# On connected algebraic subgroups of groups of birtional transformations

We want to study the group
$Bir(X) := \{f : X \dashrightarrow X \mid f : \text{birt'l}\}$

e.g. $Aut(X)$: alg. group (grp + variety)
Thm (Blanc-Furter 2013): There is no structure of an alg. grp on $Bir(\mathbb{P}^2)$.

Study of $Bir(\mathbb{P}^2)$ goes back to the 19th century (Enriques, Fano...)

Rmk: The more symmetric $X$, the more complicated $Bir(X)$.
eg. $X$: minimal model of general type
$Bir(X) = Aut(X)$: finite group

Def: • An alg. subgroup of $Bir(X)$ is an alg. group $G$ acting birt'ly on $X$.
• $G$: connected, if it is conn. as an alg. group. (c.a.s)

(m c a s)

- $G$ : maximal conn. alg. subgroup of $\mathrm{Bir}(X)$

if when $\left.\begin{array}{l}\varphi \in \mathrm{Bir}(X) \\ \varphi G \varphi^{-1} \leq H \\ H : \text{conn. alg. subg.}\end{array}\right\} \Rightarrow \varphi G \varphi^{-1} = H.$

Thm (Enriques 1893)

- Every cas of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a m.c.a.s.
- $G \leq \mathrm{Bir}(\mathbb{P}^2)$ m.c.a.s. then

  $G = PGL_3$ or $G = \mathrm{Aut}^0(\mathbb{F}_n)$ $n \neq 1$

  ↳ connected component of $\mathrm{Aut}(\mathbb{F}_n)$ containing the id.

Q: Can we give a classification of m.c.a.s.

$X = \mathbb{P}^2$ Yes Enriques

$X = C \times \mathbb{P}^1$, $g(C) \geq 1$, Yes Fong

} completes 2-dim case

$X = \mathbb{P}^3$ Yes Ummemura

Blanc-Fanelli-Terpereau.

$X = \mathbb{P}^n$ $\quad n=4$, Reduction results (Blanc-Floris)

$n \geq 4$, Examples (Floris-Z.)

Q: Is every c.a.s contained in a m.c.a.s?

$X = \mathbb{P}^2, \mathbb{P}^3$ Yes (follows from classific.)

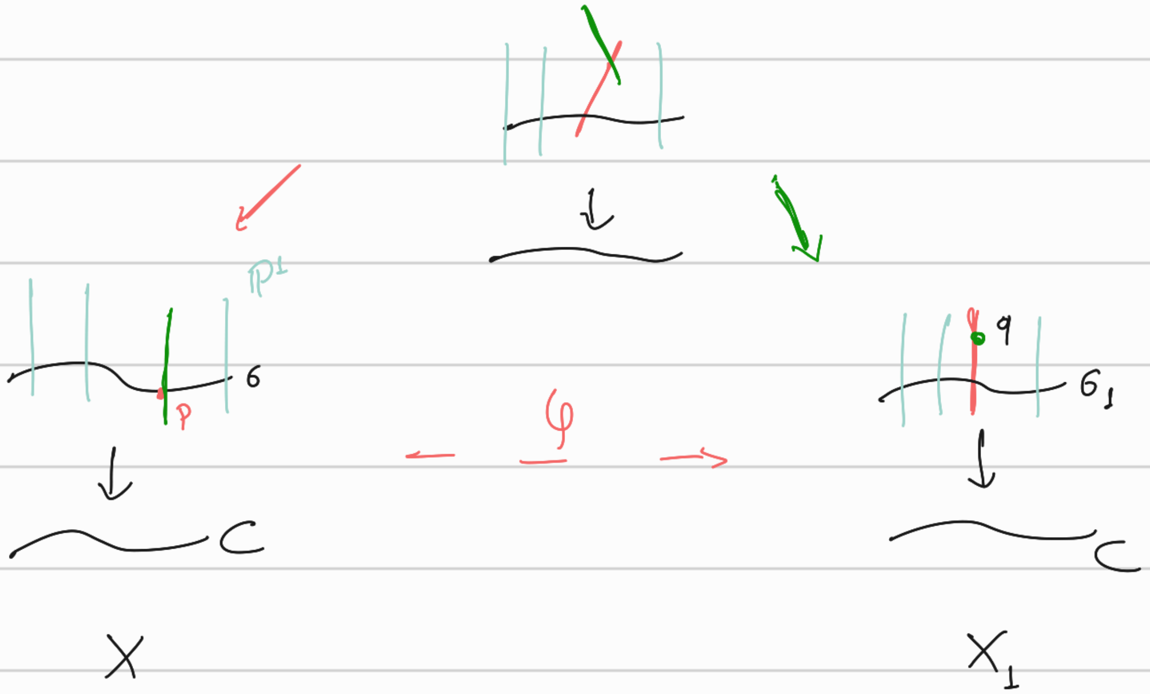
$X = \mathbb{P}^n$, $n \geq 4$ ?

Thm (Fong-Z.) $X = C \times \mathbb{P}^n$, $g(C) \geq 1$. There exist c.a.s of $\mathrm{Bir}(X)$ not contained in a maximal one.

From now on: $X$: $\mathbb{P}^1$-bundle $/ C$, $g(C) \geq 1$.

Lemma: $X \to C$ be a $\mathbb{P}^1$-bundle, $\exists \sigma \subseteq S$ with (*) $\sigma^2 \leq -2g(C)$. Then $\mathrm{Aut}^\circ(X)$ acts as follows:

- fixes all fibers.
- fixes $f \cap \sigma$
- acts transitively $f \setminus \sigma$.




$G := \mathrm{Aut}^\circ(X)$

$p$ : $G$-invariant $\Rightarrow G \curvearrowright X_1$ $\varphi$ : $G$-equiv.

$\sigma_1^2 = \sigma^2 - 1 < -2g(C)$. $\Rightarrow \mathrm{Aut}^\circ(X_1)$ does not fix $q$ $\Big\}\Rightarrow$

$G$ : fixes $q$

$\Rightarrow G \subsetneq \mathrm{Aut}^\circ(X_1)$

Claim: $\overset{G}{\overset{\shortparallel}{\mathrm{Aut}^\circ(X)}}$ is not contained in a m.c.a.s.

Proof: Suppose that $G \leq M$ : m.c.a.s.

(1) $\exists$ $\underbrace{X_m \to C : \mathbb{P}^1\text{-bundle}}_{(1.2)}$, $\underbrace{M \curvearrowright X_m}_{(1.1)}$, $X \overset{\varphi}{\dashrightarrow} X_m$

$G$-equiv.

(2) $\varphi$ : is a composition of $G$-equiv elem. transform.

$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots \dashrightarrow X_m \dashrightarrow X_{m+1}$

$X$ satisfies $\circledast$ $\Rightarrow$ $X_1$: satisfies $\circledast$ $\Rightarrow \dots \Rightarrow$ $X_m$: satisfies $\circledast$

$M \leq \mathrm{Aut}^\circ(X_m) \lneq \mathrm{Aut}^\circ(X_{m+1})$

$\Rightarrow M$: not maximal $\lightning$

(1.1): Weil's regularization thm 1955:

If $G$: c.a.s. $\mathrm{Bir}(X)$, $\exists$ $\varphi: X \dashrightarrow Y$: bir'l. $G$-equiv.

$G \curvearrowright Y$

(1.2) MMP:

$G \curvearrowright Y \rightsquigarrow$ MMP $\rightsquigarrow G \curvearrowright Z$: bir'l to $Y$, simpler (Mfs)

(2) $G$-equiv. Sark. progr. (Floris 2018)

$\varphi: Z_1 \dashrightarrow Z_2$: $G$-equiv. bir'l. $Z_i$: Mfs

$\rightsquigarrow \varphi$: composition of $G$-equiv. $\underbrace{\text{Sarkisov links}}_{\text{simple maps}}$