Log Calabi-Yau geometry and Cremona maps

I- Cremona groups and the Sarkisov Program

$\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$

A Cremona map is a bir. map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

$\mathrm{Bir}(\mathbb{P}^n) = \{ f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \text{ bir.} \}$, Cremona group

- $\mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = PGL(2, \mathbb{C})$
- $\mathrm{Bir}(\mathbb{P}^2) = \langle \mathrm{Aut}(\mathbb{P}^2), (x:y:z) \mapsto (yz:xz:xy) \rangle$

  Noether-Castelnuovo Thm.

- $\mathrm{Bir}(\mathbb{P}^n) = ?$

  For $n \geqslant 3$, there is no a reasonable presentation for this group so far

Thm. (Blanc, Lamy, Zimmermann 2019):

For $n \geqslant 3$, there exist non trivial group homomorphisms

$$\mathrm{Bir}(\mathbb{P}^n) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}.$$

From the Minimal Model Program (MMP) point of view, $\mathbb{P}^n$ together with the morphism $\mathbb{P}^n \longrightarrow \operatorname{Spec}(\mathbb{C})$ has the structure of Mori fibered space.

Rough sketch of the MMP

$$\underset{\text{terminal projective variety}}{X} \dashrightarrow \begin{cases} \cdot\ X' \text{ with } K_{X'} \text{ nef (minimal model),} \\ \qquad\qquad \text{if } \kappa(X) \geq 0 \\ \cdot\ \begin{array}{c} X \\ \downarrow \\ S \end{array} \quad \text{Mori fibered space,} \\ \qquad\qquad \text{if } \kappa(X) = -\infty \end{cases}$$

MMP holds for $\dim \leq 3$

MMP is conjectural in some cases for $\dim > 3$

Def.: A Mori fibered space (MFS) is a normal proj. var. X together with a morphism $f: X \longrightarrow S$ where $\dim X > \dim S$, st

1) $f_* \mathcal{O}_X = \mathcal{O}_S$
2) $-K_X$ is f-ample, and
3) $\rho(X/S) := \rho(X) - \rho(S) = 1$

We denote such a structure by X/S.

Thm. (Sarkisov Program - Corti 1995; Hacon, McKernan 2013): Any bir. map between MFS is a composition of finitely many Sarkisov links.

$$
\begin{tikzcd}
X = X_0 \arrow[r, dashed, "\phi_1"] \arrow[d] \arrow[rrrr, dashed, bend left, "\phi"] & X_1 \arrow[r, dashed, "\phi_2"] \arrow[d] & \cdots \arrow[r, dashed, "\phi_{m-2}"] & X_{m-1} \arrow[r, dashed, "\phi_m"] \arrow[d] & X_m = Y \arrow[d] \\
S = Y_0 & Y_1 & & Y_{m-1} & Y_m = T
\end{tikzcd}
$$

$\phi : X/S \dashrightarrow Y/T$ bir. map

$X_i \to Y_i$'s MFS

ϕ_i's are Sarkisov links

Types of Sarkisov links

Type I: dim 2

$$
\begin{tikzcd}
& \mathbb{F}_1 \arrow[dl, "Bl_p"'] \arrow[d] \\
\mathbb{P}^2 \arrow[d] & \mathbb{P}^1 \arrow[dl] \\
Spec(\mathbb{C}) &
\end{tikzcd}
$$

dim ≥ 3

$$
\begin{tikzcd}
Z \arrow[r, dashed, "\text{seq. of flips, flops, antiflips}"] \arrow[d, "\text{divisorial contraction}"'] & X' \arrow[d] \\
X \arrow[d] & S' \arrow[dl] \\
S &
\end{tikzcd}
$$

Type II: dim 2

$$\begin{tikzcd}
 & X \arrow[dl, "Bl_p"'] \arrow[dr, "\text{blow-down of the strict transform of the fiber by } p"] & \\
\mathbb{F}_n \arrow[rr, dashed] \arrow[d] & & \mathbb{F}_{n+1} \arrow[d] \\
\mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1
\end{tikzcd}$$

dim ≥ 3

$$\begin{tikzcd}
 & Z \arrow[r, dashed, "\text{seq. of flips, flops, antiflips}"] \arrow[dl, "\text{div. cont.}"'] & Z \arrow[dr, "\text{div. cont.}"] & \\
X \arrow[d] & & & X' \arrow[d] \\
S \arrow[rrr, equal] & & & S'
\end{tikzcd}$$

Type III: Inverse of type I

Type IV:

dim 2

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, squiggly, "\text{invol.}"] \arrow[d, "p_1"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "p_2"] \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^2 \arrow[dl] \\
& \mathrm{Spec}(\mathbb{C}) &
\end{tikzcd}$$

dim ≥ 3

$$\begin{tikzcd}
X \arrow[rr, dashed, "\text{seq. of flips, flops, antiflips}"] \arrow[d] & & X' \arrow[d] \\
S \arrow[dr] & & S' \arrow[dl] \\
& T &
\end{tikzcd}$$

II - Log Calabi-Yau geometry

Def.: A log Calabi-Yau pair (CY) is a lc pair (X, D) consisting of a normal proj. var. X and a reduced Weil divisor D on X st $K_X + D \sim 0$.

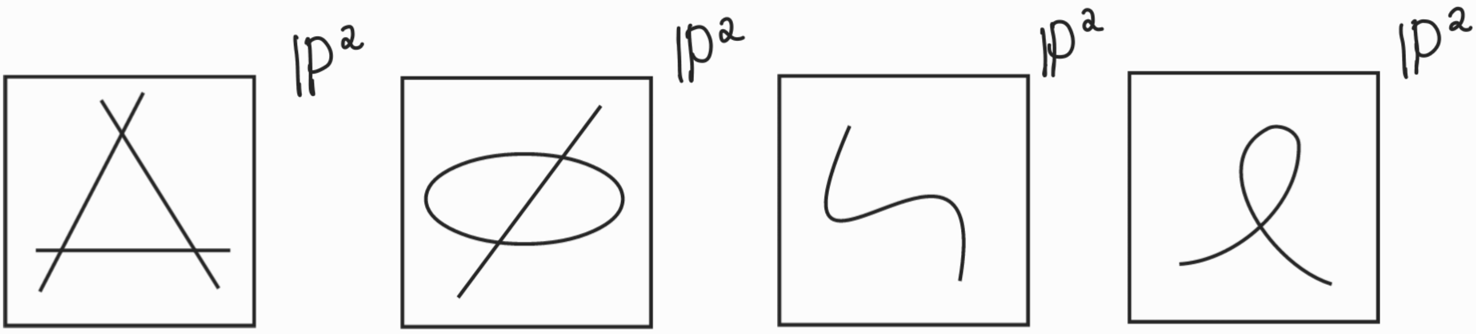
Rmk.: $n = \dim X$

(X,D) CY $\Rightarrow \exists\, \omega := \omega_{X,D} \in \Omega^n_X$ unique up to nonzero scaling st

$$D + \mathrm{div}(\omega) = 0$$

We call ω the volume form.

Example: $X = \mathbb{P}^2$



$L_1 + L_2 + L_3$	$L + C'$	C	C
3 pairwise concurrent lines	line and conic	nonsingular cubic	nodal cubic

Def.: Let (X, D_X), (Y, D_Y) be CY pairs. A bir. map $f: X \dashrightarrow Y$ is volume preserving if $f^*(\omega_{Y,D_Y}) = \lambda \omega_{X,D_X}$, for some $\lambda \in \mathbb{C}^*$, where $f^*: \Omega^n_Y \longrightarrow \Omega^n_X$ is the induced pullback by f and $n = \dim X = \dim Y$.

Rmk.: $\underset{\substack{\parallel \\ \text{group of self-vol. pres. maps}}}{\mathrm{Bir}^{vp}(X, D)} \underset{\text{subgroup}}{\subset} \mathrm{Bir}(X)$

- (X, D) CY pair

$K_X + D \sim 0 \Rightarrow -K_X = D \geq 0$

$\Rightarrow K_X$ is not pseudoeffective

$\overset{(*)}{\Rightarrow} X$ is uniruled

$\Rightarrow K(X) = -\infty$

$\Rightarrow$ output of the MMP on X is a MFS X'/S

(*) Boucksom, Demailly, Păun, Peternell

Thm. (Vol. pres. Sarkisov Program - Corti, Kaloghiros 2016): Any vol. pres. map between MF CY pairs is a composition of vol. pres. Sarkisov links.

$$\begin{tikzcd}
(X,D_X)=(X_0,D_0) \arrow[r, dashed, "\phi_1"] \arrow[d] \arrow[dd, dashed, bend left, "\phi"] & (X_1,D_1) \arrow[r, dashed, "\phi_2"] \arrow[d] & \cdots \arrow[dd, dashed, "\phi_{m-2}"] \\
S = Y_0 & Y_1 & \\
(Y,D_Y)=(X_m,D_m) \arrow[d] & & (X_{m-1},D_{m-1}) \arrow[ll, dashed, "\phi_{m-1}"'] \arrow[d] \\
Y_m & & Y_{m-1}
\end{tikzcd}$$

$\phi : (X,D_X)/S \dashrightarrow (Y,D_Y)/T$ vol. pres. map

$(X_i,D_i) \longrightarrow Y_i$ MF CY pairs

ϕ_i's are vol. pres. Sarkisov links

Def.: Let X be a proj. var. and $\mathrm{Bir}(X)$ be its group of bir. aut.. Given $Y \subset X$ an (irred.) subvar., the decomposition group of Y in $\mathrm{Bir}(X)$ is the group

$$\mathrm{Bir}(X,Y) = \{\varphi \in \mathrm{Bir}(X) \mid \varphi(Y) \subset Y, \quad \varphi|_Y : Y \dashrightarrow Y \text{ is bir.}\}$$

The inertia group of Y in $\mathrm{Bir}(X)$ is the group

$$\{\varphi \in \mathrm{Bir}(X,Y) \mid \varphi|_Y = \mathrm{Id}_Y \text{ as bir. map}\}$$

Notation: When $X = \mathbb{P}^n$, we denote such groups by $\mathrm{Dec}(Y)$ and $\mathrm{Ine}(Y)$, respectively.

Prop. (Araujo, Corti, Massarenti 2023): (X,D) CY pair with D a prime divisor. $\mathrm{Bir}^{vp}(X,D) = \mathrm{Bir}(X,D)$ provided the pair (X,D) is canonical

II - The 2-dimensional case

$C \subset \mathbb{P}^2$ nonsingular cubic

$(\mathbb{P}^2, C)$ is a canonical CY pair

Thm (Pan 2007): Let $C \subset \mathbb{P}^2$ be an irred., nonsing. and nonrational curve and suppose there is $\phi \in \mathrm{Dec}(C) \setminus PGL(3, \mathbb{C})$. Then $\deg(C) = 3$ and $Bs(\phi) \subset C$, where $Bs(\phi)$ denotes the set of proper base points of ϕ.

Notation: $Bs(\phi) :=$ proper base locus of a rat. map $\phi : X \dashrightarrow Y$ between proj. var.

$\underline{Bs}(\phi) :=$ full base locus, including the

Lemma (- 2023): Let $C \subset \mathbb{P}^2$ be a nonsing. cubic. Consider $\phi \in \mathrm{Dec}(C) \setminus PGL(3, \mathbb{C})$. Then $\underline{Bs}(\phi) \subset C$.

Thm (- 2023): Let $C \subset \mathbb{P}^2$ be a nonsing. cubic. The standard Sarkisov Program applied to an elem. of $\mathrm{Dec}(C)$ is automatically vol. preserving.

IV - The 3-dimensional case

(X, D_X) CY pair

$\mathrm{coreg}(X, D_X) \in \{0, 1, \dots, \dim X\}$

The coreg. is the most important discrete vol. pres. invariant.

Ducat classified all pairs of the form $(\mathbb{P}^3, D)$ with coreg $\leq 0, 1$ up to vol. pres. equivalence.

- $\mathrm{coreg}(\mathbb{P}^3, D) = 2 \iff (\mathbb{P}^3, D)$ is canonical
  $\iff D$ is an irred. normal quartic surface with can. singularities

- Strict can. sing. for surfaces $\overset{1-1}{\Leftrightarrow}$ ADE Dynkin diagrams

Problem: Given the type of can. sing. at P at a point $P \in D$, to determine for which weights $(1, a, b)$ the Kawakita blowup $\pi : (X, \tilde{D}) \to (\mathbb{P}^3, D)$ at P with weights $(1, a, b)$ is vol. preserving.

Thm (Guerreiro 2022): Let $\varphi: X \longrightarrow \mathbb{P}^3$ be the toric $(1,a,b)$-weighted blowup of a point. Then φ initiates a Sarkisov link of $\mathbb{P}^3$

$$\iff (a,b) \in \{(1,1),(1,2),(2,3),(2,5)\},$$

up to permutation of a and b.

Thm (- 2023): Let $(\mathbb{P}^3, D)$ be a CY of cor. 2 and $\pi: (X, \tilde{D}) \longrightarrow (\mathbb{P}^3, D)$ be a vol. pres. toric $(1,a,b)$-weighted blowup of a torus invariant pt.. Then this pt. is a sing. of D and, up to perm., the only possibilities for the weights initiating a vol. pres. Sarkisov link are described in the following table.

type of sing.	possible vol. pres. weights
A_1	$(1,1,1)$
A_2	$(1,1,1)$, $(1,1,2)$
A_3	$(1,1,1)$, $(1,1,2)$
A_4	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
A_5	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
$A_{\geq 6}$	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$, $(1,2,5)$

D_4	$(1,1,1), (1,1,2)$
$D_{\geq 5}$	$(1,1,1), (1,1,2), (1,2,3)$
E_6	$(1,1,1), (1,1,2), (1,2,3)$
E_7	$(1,1,1), (1,1,2), (1,2,3)$
E_8	$(1,1,1), (1,1,2), (1,2,3)$