Donaldson-Thomas invariants

– of –

Threefold Flops

Quick reminder: flops in the MMP

MMP: given sm. proj. variety X find birational modification

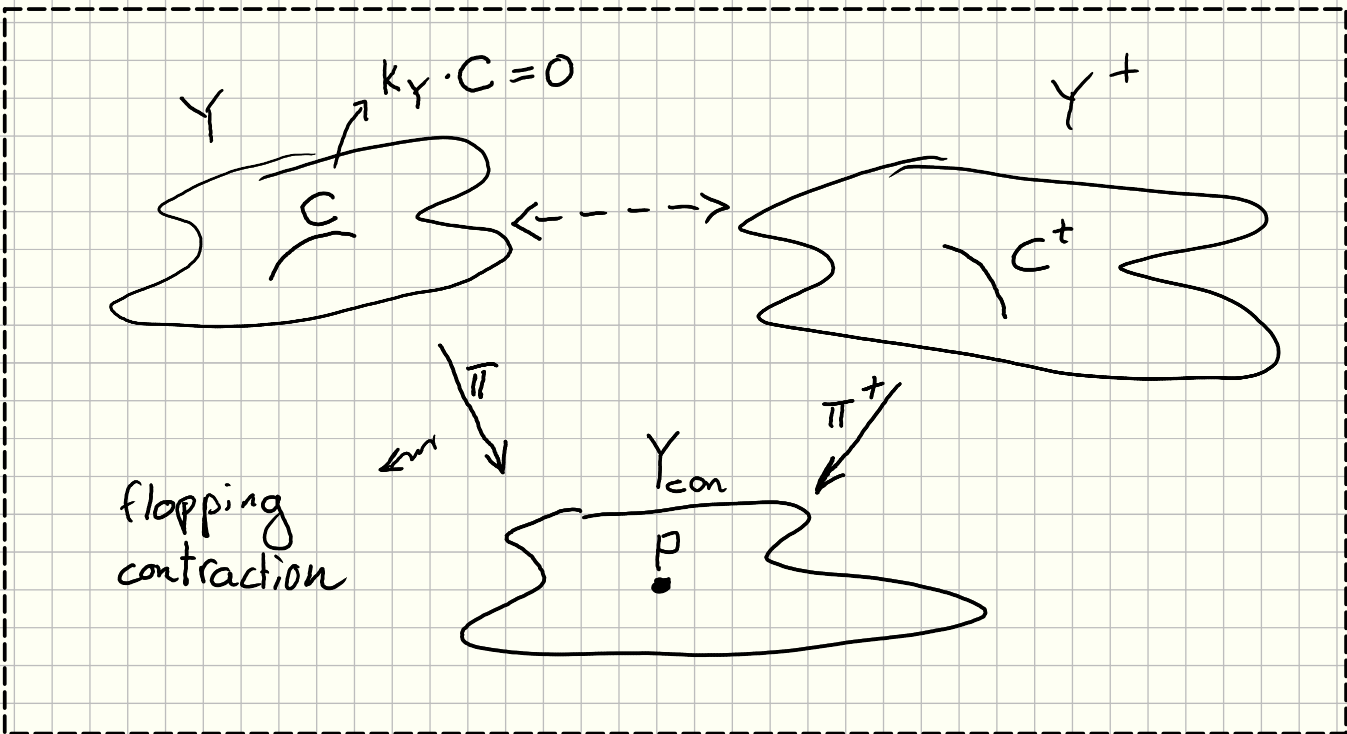
$$X \dashrightarrow \cdots\cdots \dashrightarrow Y$$

s.t. $K_Y \cdot C \geq 0$ for $C \subset Y$ rational curve (nef)

Such a Y is a <u>minimal model</u> of X

dim $= 2$: minimal models are unique

dim $= 3$: finitely many connected via flops

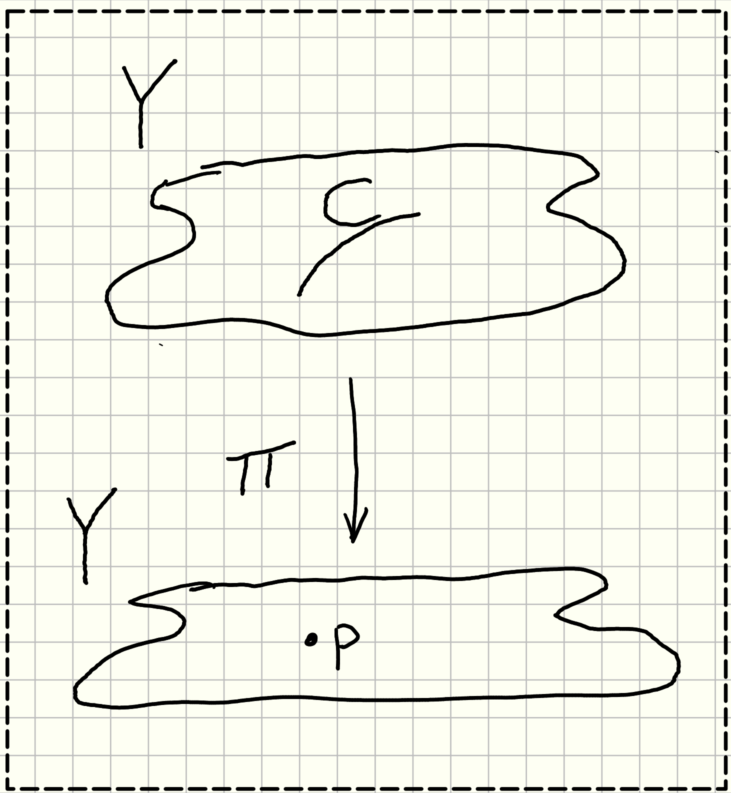


- Rich local geometry $\rightsquigarrow$ singularity $p \in Y_{con}$
- Intricate deformation theory (esp. noncommutative)
- Examples of local Calabi-Yau threefolds

Local geometry & invariants

This talk: Y smooth nbhd of a flopping curve $C \simeq \mathbb{P}^1$

affine base $Y_{con} = \operatorname{Spec} R$



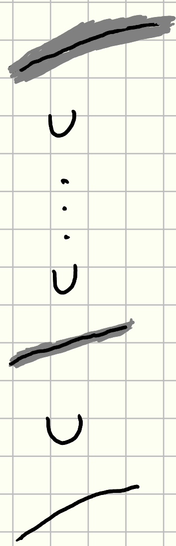
chain of subschemes:

$$\pi^{-1}(p) =: \ell C$$

schemey fibre

$$\ell C \supset \cdots \supset 2C \supset C$$

our curve $\simeq \mathbb{P}^1$



$\rightsquigarrow$ length classification (Katz – Morrison '93)

$$\ell = \underset{\text{easy}}{1},\ \underbrace{2,\ 3,\ 4,\ 5 \text{ or } 6}_{\text{complicated}}$$

$\boxed{\ell = 1}$ completely classified by Reid ('83)

$\rightsquigarrow$ family in a single integer invariant

What about $\ell > 1$?

Generalising Reid's invariant:

string theory prediction

$\rightsquigarrow$

Gromov-Witten theory of Y controlled by <u>Gopakumar-Vafa invariants</u>:

$$\boxed{(n_1, n_2, \dots, n_\ell) \in \mathbb{Z}^\ell}$$

'curve counts' of the curves $C, 2C, \dots, \ell C$

<u>Fact</u>: For $\ell = 1$, single invariant n_1 coincides with Reid's invariant
$\rightsquigarrow$ complete classifying invariant

<u>However</u>: GV invariants do <u>not</u> classify flops for $\ell \geq 2$ (Brown-Wemyss '17)
$\Rightarrow$ stronger invariants required

What further generalisation can we make?

Shown by Katz ('08):

GV invariants are <u>Donaldson-Thomas invariants</u>:

$$\boxed{n_i = \text{"virtual count of sheaves } \mathcal{F} \text{ supp. on } C \text{ w/ class } [\mathcal{F}] = [\mathcal{O}_{iC}] \in K_0(\operatorname{coh} Y)\text{"}}$$

<u>Plan</u>: further investigate the DT theory of flops

Using DT theory, generalise in two ways:

1. Extend Following method of Joyce-Song:

for every $(r,d) \in K_0(\mathrm{coh}_{cs} Y) \simeq K_0(\mathrm{coh}\, \mathbb{P}^1)$ define invariants

compactly supported

$DT_{(r,d)}$ = "Virtual count of points on moduli space of objs. in $D^b(\mathrm{coh}_c Y)$ with class (r,d)"

Recovers $n_i = DT_{[\mathcal{O}_{C_i}]}$, but also other invariants

2. Refine As predicted by Kontsevich-Soibelman:

Invariants are "motivic" $\rightsquigarrow$ behave like a cs cohomology theory

$\Rightarrow$ $DT_{(r,d)}$ can be calculated as motive/Hodge-structure/etc.

! requires extra data that captures deformation theory of semistable objects

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First calculation of this type for $\ell = 1$ by

(Morrison-Mozgovoy-Nagao-Szendrői, Davison-Meinhardt)

<u>However</u>: moduli spaces not as well understood for $\ell \geq 2$

$\Rightarrow$ We need to find the stability conditions on $D^b(\mathrm{coh}_{cs} Y)$

<u>Also:</u> Need to get a grip on the deformation theory to find refined invariants

Stability conditions

<u>Problem</u>: want to count moduli of objs, but $D^b(\mathrm{coh}_{cs} Y)$ is complicated...

<u>Idea</u>: filter by a more manageable set of <u>semistable objects</u>

<u>Bridgeland-stability condition</u>: $\begin{cases} \mathcal{A} \subset D^b(\mathrm{coh}_{cs} Y) & \text{abelian heart of a t-struct.} \\ \text{for } \phi \in (0,1] & \mathcal{S}_\phi \subset \mathcal{A} \text{ semistables of phase } \phi \end{cases}$

<u>Harder-Narasimhan property</u>: Every obj $M \in \mathcal{A}$ has a unique filt.

$M \supset M_1 \supset M_2 \supset \ldots \supset M_n$ s.t. M_i / M_{i+1} ss w/ ordered phases ϕ

$\rightsquigarrow$ moduli space stratified by HN-type (Reineke)

$$\mathcal{M} = \bigsqcup_{\phi_1 < \cdots < \phi_n} \mathcal{M}_{\phi_1, \ldots, \phi_n}$$

moduli stack of objs

stack of objs s.t. HN filt has phases ϕ_i

In DT theory: Obtain a generating expression (Kontsevich-Soibelman)

$$\sum_\delta DT_\delta \cdot t^\delta = \mathrm{Sym}\Big(\sum_{\substack{\delta \text{ of ss} \\ \text{objects}}} BPS_\delta \cdot t^\delta \Big)$$

$\Rightarrow$ If we know the classes of ss-objects & their modulispaces we win!

<u>Require</u>: a systematic way of understanding stab-conditions.

Tilting theory & stability

Recall: a vector bundle $\mathcal{T}$ on Y is tilting if
- $R\mathrm{End}(\mathcal{T}) = \mathrm{End}_Y(\mathcal{T})$ (no self-extensions)
- $\mathcal{T}$ generates $D^b(\mathrm{coh}\, Y)$

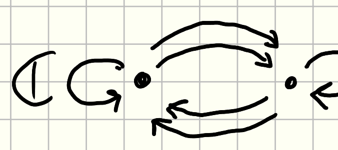
In this case: derived equivalence

$$\begin{tikzcd} D^b(\mathrm{coh}_{cs} Y) \arrow[r, shift left, "R\mathrm{Hom}(\mathcal{T},-)"] & D^b(\mathrm{fdmod}\, \underbrace{\mathrm{End}_Y(\mathcal{T})}_{\Lambda}) \arrow[l, shift left, "-\otimes \mathcal{T}"] \end{tikzcd}$$

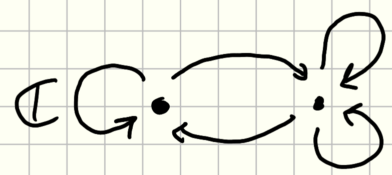
<u>Benefit</u>: $\mathrm{fdmod}\,\Lambda$ is a length-category (Jordan-Hölder property)
$\Rightarrow$ easier to find filtrations, hence stab-cond

What does the tilting theory of flops look like?

(Van den Bergh '04) $\rightsquigarrow$ flops admit a tilting bundle $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$

Λ:  /relns

$\ell = 1$

 /relns

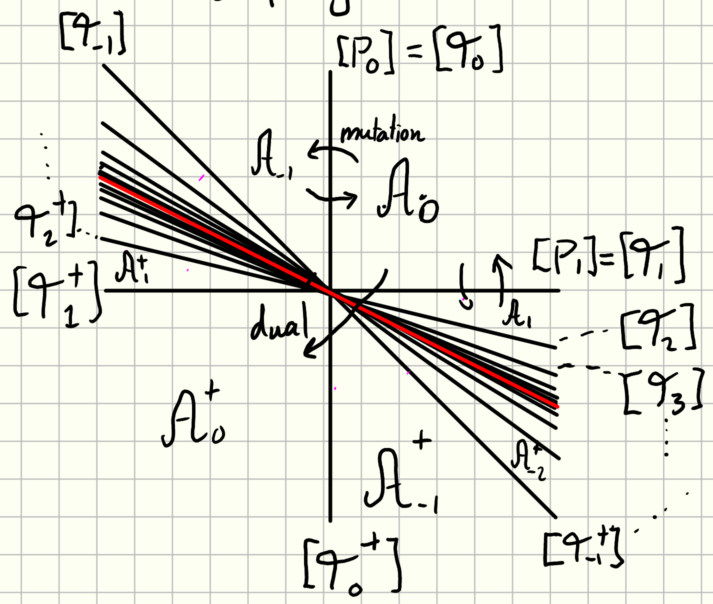
$\ell \geq 2$

multi-curve flops: additional vertices & arrows

Full theory: many tilting bundles (Iyama-Wemyss, Hirano-Wemyss, Donovan-Wemyss & others)

Wall-and-chamber structure (example $\ell = 2$)

$K_0(\mathrm{proj}\,\Lambda) \simeq K_0(Y)$



Every chamber:

- tilting bundle $q_i \oplus q_{i+1}$ or $q_i^+ \oplus q_{i+1}^+$
- quiver algebra $\Lambda_i = \mathrm{End}(q_i \oplus q_{i+1})$
- abelian heart $\mathcal{A}_i = \mathrm{fdmod}\,\Lambda_i$
- neighbouring chambers related via <u>mutation</u>
- opposing chambers dual

<u>Linear stability conditions</u>: the functor $R\mathrm{Hom}(-,-)$ induces a pairing

$$(-,-) : \underset{\substack{\wr\wr \\ K_0(Y)}}{K_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}}} \underset{\mathbb{Z}}{\otimes} \underset{\substack{\wr\wr \\ K_0(\mathrm{coh}_{cs} Y)}}{K_0(\mathrm{fdmod}\,\Lambda)} \longrightarrow K_0(\mathbb{C})_{\mathbb{R}} \simeq \mathbb{R}$$

$\rightsquigarrow$ every $\theta \in K_0(\mathrm{proj}\,\Lambda)_{\mathbb{R}}$ induces a phase function

$$\Theta : K_0(\mathrm{fdmod}\,\Lambda) \xrightarrow{(\theta,-)} \mathbb{R} \sim (0,1)$$

$\Rightarrow$ stability conditions with heart $\mathcal{A} = \mathrm{fdmod}\,\Lambda$ and

$$\mathcal{S}_\phi = \{ M \in \mathrm{fdmod}\,\Lambda \mid \Theta([M]) = \phi,\ N \subset M \Rightarrow \Theta([N]) \leq \Theta([M]) \}$$

- have the HN property
- Every $\mathcal{S}_\phi$ generated by 'stable' objs (Relative JH)

Can already find 2 stable modules in each heart:

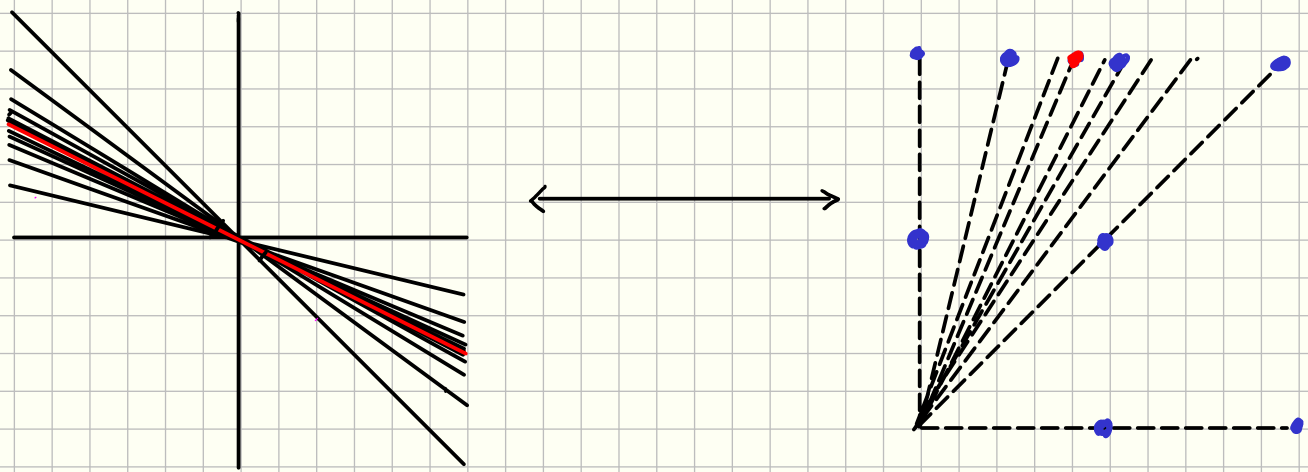
- simples $S_i, S_{i+1} \in \mathrm{fdmod}\,\Lambda_i$
- proj. covers $\left.\begin{array}{l} P_i = \mathrm{Hom}(T_i \oplus T_{i+1}, T_i) \\ P_{i+1} = \mathrm{Hom}(T_i \oplus T_{i+1}, T_{i+1}) \end{array}\right\} \in \mathrm{proj}\,\Lambda_i \rightsquigarrow$ span walls
- $([P_i],[S_{i+1}]) = ([P_{i+1}],[S_i]) = 0 \rightsquigarrow$ orthogonal to walls

By tilting between chambers $\mathcal{A}_j \longleftrightarrow \mathcal{A}_0$

get modules: $S_j \longleftrightarrow M_j$

Thm 1: Linear stability induces a duality between stables & walls

$K_0(\mathrm{proj}\,\Lambda) = K_0(Y)_{\mathbb{R}}$ $\qquad\qquad$ $K_0(\mathrm{fdmod}\,\Lambda) = K_0(\mathrm{coh}_{cs} Y)$



I.e. every stable module in $\mathcal{A}_0$ is obtained as some M_i

or has a class dual to - accumulation ray

- Also works for other flops ($\ell \geq 2$, multi-curve, ...)

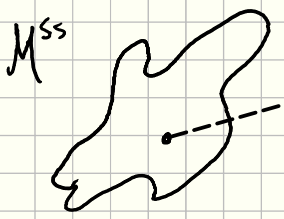
$\Rightarrow$ All stable modules are images of some simple S_i

classified for all simple flops (Donovan–Wemyss '19)

$\ell \geq 1$	$\ell \geq 2$	$\ell \geq 3$
$\mathcal{O}_p$ $p \in C$, $\mathcal{O}_C(n)[m]$	$\mathcal{O}_{2C}(n)[m]$	$\mathcal{O}_{KC}(n)[m]$ + more exotic stuff

Refined Donaldson-Thomas invariants

Refinement depends on a potential:



$\mathcal{F} \in \mathrm{coh}_{cs} Y \rightsquigarrow$

deformations captured by A_∞-alg

$E_{\mathcal{F}} := \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$

$\underbrace{+ \langle -,- \rangle \quad E_{\mathcal{F}} \otimes E_{\mathcal{F}} \to \mathbb{C}[-3] \quad \text{inner product}}$

$\downarrow$

induces a potential $W_{\mathcal{F}}$

Compatible with tilting?

✓ Tilting functors lift to A_∞-enhancement

? Inner product depends on CY structure $HH_3(\mathrm{coh}_{cs} Y) \to \mathbb{C}$

As in K-theory, pairing between $HH_3(Y)$ and $HH_3(\mathrm{coh}_{cs} Y)^*$

$$\left\{ \begin{matrix} \text{Calabi-Yau} \\ \text{volume } HH_3(Y) \end{matrix} \right\} \times \left\{ \begin{matrix} \text{Calabi-Yau} \\ \text{pairing } D^b(\mathrm{coh}_{cs} Y) \end{matrix} \right\} \longrightarrow \mathbb{C}$$

Invariant under tilting!

Thm 2: Let $\mathcal{F}$ be a tilting functor preserving CY-volume:

$$HH_3(\mathcal{F}) \circlearrowright HH_3(Y) \quad \text{is trivial} \quad (\text{up to scalar } \lambda \in \mathbb{C}^\times)$$

Then $\mathcal{F}$ preserves potentials & hence refined invariants:

$$W_{\mathcal{F}(N)} \sim W_N \cdot \lambda \quad , \quad DT_{[\mathcal{F}(N)]} = DT_{[N]} \quad \text{in refined theory}$$

Results for $\ell=2$ flops

Can test these methods out on flops of length 2:

$\rightsquigarrow$ family $Y_{a,b}$ $\quad a,b\in\mathbb{N}$ $\quad$ (Laufer '81, Pinkham '83
Brown-Wemyss '17, Kawamata '20)

Apply the theorems:

__Thm 1__ $\Rightarrow$ Stable objects are points & curves

$$\boxed{\mathcal{O}_p\,, \quad \mathcal{O}_C(n)[m]\,, \quad \mathcal{O}_{2C}(n)[m]}$$

__Thm 2__ $\Rightarrow$ the functors $\mathcal{O}_Y(n)\otimes(-)[m]$ preserves DT invariants

$\Rightarrow$ The DT theory is captured by the generating function:

$$\boxed{\Phi(t) = \mathrm{Sym}\left(\underbrace{\sum_k \mathrm{BPS}_{k[pt]}\cdot t^{k\cdot(0,1)}}_{\text{rank } 0} + \sum_{k,n}\Big(\underbrace{\mathrm{BPS}_{k[C]}\cdot t^{\pm k(1,n)}}_{\text{rank } 1} + \underbrace{\mathrm{BPS}_{k[2C]}\cdot t^{\pm k(2,2n-1)}}_{\text{rank } 2}\Big)\right)}$$

point counts

curve counts

__Takeaway__: all sheaf counting captured by point counts & curve counts (GV)

__Moreover__: direct calculation shows: (Verified up to a certain level of refinement)
(Hodge-theoretic)

$$\boxed{\mathrm{BPS}_{k[2C]} = \mathrm{BPS}_{2k[C]}\,, \qquad \mathrm{BPS}_{k[C]} = 0 \text{ for } k>\ell}$$

<u>Conclusion</u>: DT theory controlled by finitely many invariants

$$\mathrm{BPS}_{[pt]},\ \mathrm{BPS}_{[C]},\ \mathrm{BPS}_{[2C]}$$

same for all flops

GV invariants (n_1, n_2)

$\Rightarrow$ strong rationality conjecture (Pandaripande-Thomas) for refined invariants

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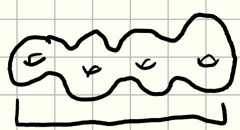
<u>What about refinements?</u> for family $Y_{a,b}$ of flops:

$$\boxed{\begin{aligned} \mathrm{BPS}_{[pt]} &= [Y_{a,b}]_{virt} \quad \rightsquigarrow \text{motivic count of points} \\ \mathrm{BPS}_{[C]} &= \mathbb{L}^{-1}\big(1 - [\ \cdot\ \circlearrowleft \mathbb{Z}/4a\mathbb{Z}]\big) + 2 \\ \mathrm{BPS}_{[2C]} &= \mathbb{L}^{-\frac{1}{2}}\big(1 - [\mu_a \circlearrowleft \mathbb{Z}/a\mathbb{Z}]\big) \end{aligned}}$$



genus a

$\rightsquigarrow$ depends only on one of the params but have noniso flops $Y_{(a,b)} \not\cong Y_{(a,b')}$ with same invariants

<u>Conclusion</u>: Refined DT invariants still don't determine flops

<u>Alternatives</u>: noncommutative deformation theory

$\rightsquigarrow$ contraction algebra invariant (Donovan-Wemyss)