MATH1003
ASSIGNMENT 6
ANSWERS

1. Let $g(x) + x\sin g(x) = x^2$, and $g(1) = 0$. Differentiating both sides we obtain:

$$g'(x) + \sin g(x) + xg'(x)\cos g(x) = 2x.$$

Setting $x = 1$ gives:

$$g'(1) + \sin 0 + g'(1) = 2.$$

Rearranging we see that $g'(1) = 1$. Differentiating a second time gives:

$$\begin{aligned} g''(x) + g'(x)\cos g(x) + g'(x)\cos g(x) + xg''(x)\cos g(x) \\ -x(g'(x))^2 \sin g(x) = 2. \end{aligned}$$

We see that:

$$g''(1) + 1 + 1 + g''(1) - 0 = 2$$

and so $g''(1) = 0$.

2. (i) Differentiating both sides implicitly gives:

$$\begin{aligned} \frac{dy}{dx}\sin x^2 + 2xy\cos x^2 &= \sin y^2 + 2xy\frac{dy}{dx}\cos y^2 \\ \Rightarrow \quad \frac{dy}{dx}\sin x^2 - 2xy\frac{dy}{dx}\cos y^2 &= \sin y^2 - 2xy\cos x^2 \\ \Rightarrow \quad (\sin x^2 - 2xy\cos y^2)\frac{dy}{dx} &= \sin y^2 - 2xy\cos x^2 \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{\sin y^2 - 2xy\cos x^2}{\sin x^2 - 2xy\cos y^2} \end{aligned}$$

(ii) Recall that:

$$\frac{d}{dz}\cot z = -\csc^2 z.$$

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Hence, via implicit differentiation of both sides:

$$\begin{aligned}
y + x\frac{dy}{dx} &= -\Big(y + x\frac{dy}{dx}\Big)\csc^2(xy) \\
\Rightarrow \qquad x\frac{dy}{dx} + x\frac{dy}{dx}\csc^2(xy) &= -y\csc^2(xy) - y \\
\Rightarrow \qquad x\Big(1+\csc^2(xy)\Big)\frac{dy}{dx} &= -y\Big(1+\csc^2(xy)\Big) \\
\Rightarrow \qquad \frac{dy}{dx} &= -\frac{y}{x}.
\end{aligned}$$

(iii) First we consider the right-hand side. Let $u = xy^2$. Then $\sin(xy^2) = \sin u$, and by the Chain Rule:

$$\begin{aligned}
\frac{d}{dx}\sin(xy^2) &= \frac{d}{du}\sin u \times \frac{du}{dx} \\
&= \cos u \times \Big(y^2 + 2xy\frac{dy}{dx}\Big) \\
&= \cos(xy^2)\Big(y^2 + 2xy\frac{dy}{dx}\Big).
\end{aligned}$$

Hence:

$$\begin{aligned}
1 + x &= \sin(xy^2) \\
\Rightarrow \qquad 1 &= \cos(xy^2)\Big(y^2 + 2xy\frac{dy}{dx}\Big) \\
\Rightarrow \qquad 2xy\cos(xy^2)\frac{dy}{dx} &= 1 - y^2\cos(xy^2) \\
\Rightarrow \qquad \frac{dy}{dx} &= \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)} \\
&= \frac{1}{2xy}\sec(xy^2) - \frac{y}{2x}.
\end{aligned}$$

3. (i) Let $y = \tan^{-1} x$. Then $\tan y = x$. Differentiating both sides we obtain:

$$\begin{aligned}
\sec^2 y\frac{dy}{dx} &= 1 \\
\Rightarrow \qquad \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

(ii) Let $u = \sqrt{x}$, so that $y = \tan^{-1} u$. By the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Hence, using our answer to (i), we obtain:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2\sqrt{x}} \times \frac{1}{1+u^2} \\ &= \frac{1}{2\sqrt{x}} \times \frac{1}{1+(\sqrt{x})^2} \\ &= \frac{1}{2\sqrt{x}(1+x)}.\end{aligned}$$

(iii) First we calculate the derivative of $\cos^{-1} x$. Let $u = \cos^{-1} x$. Then $\cos u = x$. Differentiating gives:

$$\begin{aligned} -\sin u \frac{du}{dx} &= 1 \\ \Rightarrow \qquad \frac{du}{dx} &= -\frac{1}{\sin u} \\ &= -\frac{1}{\sqrt{1-\cos^2 u}} \\ &= -\frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

Using our answer to (i) we see that:

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}}.$$

4. From the graph of $x^2 - xy + y^2 = 3$ we see that the minimum and maximum values of y occur when the tangent is parallel to the x-axis. The minimum and maximum values of x occur when the tangent is parallel to the y-axis. Differentiating implicitly we find that:

$$\begin{aligned} 2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\ \Rightarrow (2y - x)\frac{dy}{dx} &= y - 2x \\ \Rightarrow \frac{dy}{dx} &= \frac{y-2x}{2y-x}.\end{aligned}$$

The tangent is parallel to the x-axis when $y - 2x = 0$, i.e. when $y = 2x$. Substituting this back into the equation of the tilted ellipse gives:

$$\begin{aligned} x^2 - x(2x) + (2x)^2 &= 3 \\ \Rightarrow x^2 - 2x^2 + 4x^2 &= 3 \\ \Rightarrow x^2 &= 1 \\ \Rightarrow x &= \pm 1. \end{aligned}$$

Thus the minimum value of y is -2 and the maximum value of y is 2.

The tangent is parallel to the y-axis when $2y - x = 0$, i.e. when $x = 2y$. Substituting this back into the equation gives:

$$\begin{aligned} (2y)^2 - (2y)y + y^2 &= 3 \\ \Rightarrow y &= \pm 1. \end{aligned}$$

We see that the minimum value of x is -2 and the maximum value of x is 2.

5. Differentiating $(x^2 + y^2)^2 = 2(x^2 - y^2)$ gives:

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 4\left(x - y\frac{dy}{dx}\right).$$

Setting $\frac{dy}{dx} = 0$, we obtain:

$$4x(x^2 + y^2) = 4x.$$

Hence either $x = 0$ or $x^2 + y^2 = 1$. Let us consider the second possibility.

Substituting into the equation of the leminscate we obtain:

$$\begin{aligned} 1^2 &= 2(x^2 - (1 - x^2)) \\ \Rightarrow \frac{1}{2} &= 2x^2 - 1 \\ \Rightarrow x^2 &= \frac{3}{4} \\ \Rightarrow x &= \pm\frac{\sqrt{3}}{2}. \end{aligned}$$

When $x^2 = \frac{3}{4}$ we see that $y^2 = 1 - \frac{3}{4} = \frac{1}{4}$.

We have found that $\frac{dy}{dx} = 0$ at *five* points: when (x, y) is equal to

$$(0,0), \quad \Big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big), \quad \Big(\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big), \quad \Big(-\frac{\sqrt{3}}{2}, \frac{1}{2}\Big), \quad \Big(\frac{\sqrt{3}}{2}, \frac{1}{2}\Big).$$

Consulting the graph we see that we have a slight problem. There should be only *four* points where the tangent line is parallel to the x-axis. Somehow the extra point $(0, 0)$ has appeared.

This is because the graph is rather unusual at $(0, 0)$. Close to the origin the graph looks like an $\times$. It is what we call a *singularity*. It is because of this singularity that we are getting our extra point; we should simply ignore the solution $(0, 0)$ as not relevant to the answer.